Paul A. Smith

1900–1980

BIOGRAPHICAL *Memoirs*

A Biographical Memoir by
John W. Morgan

NATIONAL ACADEMY OF SCIENCES

PAUL ALTHAUS **SMITH**

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By John W. Morgan

Paul Althaus Smith was born in Lebanon, NH, and received an undergraduate degree from Dartmouth College in 1921. He went to the University of Kansas to study mathematics with Solomon Lefschetz and moved with his mentor to Princeton University, where Smith continued to work under Lefschetz's direction and received his Ph.D. from Princeton in 1925. His thesis was titled "Approximation of Curves and Surfaces by Algebraic Curves and Surfaces."

After a year as fellow at Harvard University in 1926–1927, Smith joined the mathematics faculty of Columbia University and Barnard College in 1927 as an instructor; he rose through the ranks, becoming a full professor in 1940 and the Davies Professor of Mathematics in 1951. He served as department chair from 1945 to 1951 and again in 1956–1957 and 1961. Smith retired in 1968 and was Davies Professor Emeritus until his death in 1980, though he also continued to teach at Columbia as a special lecturer in mathematics from 1968 until 1971. He had six Ph.D. students and a total of 97 direct Ph.D. mathematical descendants. Smith was elected to the National Academy of Sciences in 1947 and received an honorary degree, Doctor of Science, from Columbia in 1973.

Smith served the National Academy of Sciences in various ways—as chair of the Mathematics Division (1955–1957), as chair of the Nominating Committee (1957–1958), as a member of the Committee of Scientific Conferences (1951–1955), and as chair of that committee (1952–1955). He was also a member in 1961 of the Committee on the Marsh Fund of grants-in-aid to researchers, which at that time was administered by the Academy. Smith served the American mathematical community as a member of the Committee on Mathematics Advisory to the Office of Naval Research in 1951, as a member of the U.S. National Committee for Mathematics from 1953 until 1957, as treasurer of the American Mathematical Society (AMS) in 1937, and as an editor of the Bulletin of the AMS from 1937 until 1946.

In 1935, Smith married Suzanne Bloch, a well-known musical performer and daughter of the composer Ernst Bloch. They had two sons, Matthew and Anthony. Throughout his life, Smith valued and was a friend to artisans, whether they worked with mathematical objects or physical objects. He made medieval musical instruments from scratch, many of which were played by his wife. With his own hands, he build a house in Vermont, in his beloved New England, on the top of a hill with a beautiful view. He not only chose the location and built the house, he also cleared a path for the road leading to the house and cleared trees from around the house to provide the view.

The citation for the honorary degree that Smith received from Columbia reads as follows:

> *A New Hampshire Yankee, educated at Dartmouth and Princeton, you came to the Columbia-Barnard campus in 1927 and served the university and the American mathematical community for half a century. A pioneer in topology, you created the profound theory of periodic transformations, which greatly influenced the development of mathematics and placed you in the select group of American mathematicians who changed the United States from a mathematical province to a leading mathematical nation.*
>
> *A bold and original thinker, you are as much at home with manual skills and music as you are with concepts and logical operations. A creative mathematician, an accomplished musician, and a builder of ancient musical instruments, you have added lustre, style, and grace to our community.*

Overview of Smith's work

Starting in the earliest days of his career and continuing for the rest of his life, Smith studied topological spaces and their topological symmetries—in other words, he studied group actions on topological spaces. When Smith received his Ph.D. degree, topology was a young subfield of mathematics. Some of its foundations had been laid in the second half of the 19th century, but topology as an independent field was established through the work of Henri Poincaré around the turn of the century and into the first decade of the 20th century. During that period the basics of point-set topology were established and homology theory was being studied in a systematic way, but they had not yet received the axiomatic codification that came later. Smith's work in the 1930s and '40s was some of the very first to consider the question of symmetries for topological spaces.

The idea that symmetry is an important aspect of any topological, geometric, or physical system is now widely recognized. Whole fields have arisen around the study of symmetry. In mathematics there are areas such as equivariant cohomology, equivariant index theorems, and fixed point theory and localization in cohomology. In physics, the ideas of gauge theories and symmetry are fundamental. Many of these developments have roots in Paul Smith's work. Even more important than any particular result or question of his was Smith's belief, as demonstrated by his lifelong devotion to the study, that symmetry in the topological and geometric setting was a rich topic worthy of serious investigation.

Smith's gift, his genius, was to home in on the very best type of problems and questions. These were always simple to state, often appearing with hindsight to be the obvious question to ask or the obvious result for which to aim. Yet invariably they turned out to be of profound and long-lasting importance; several are now considered landmarks of the whole subject of group actions on spaces.

Smith's work often involved very nice spaces—for example, Euclidean space or the sphere—but his results were formulated for a much more general class of spaces. Whatever the space under consideration, he always dealt with symmetries without any extra assumed properties beyond continuity. This meant that he had to confront difficult questions of local pathologies of the symmetries, and he relied heavily on the then-recent results in point set topology developed by Eduard Cech, Pavel Urysohn, Karl Menger, and James Alexander. Today, these local pathologies are not the focus of attention, and normally one treats symmetries that preserve a smooth—i.e., differentiable—structure on the space. Nevertheless, even though he was working in a technically much more challenging context, the results that Smith obtained and the questions that he asked are nontrivial and fundamental even in these more structured environments. His sense of mathematics led him to the essential features.

The Hilbert-Smith Conjecture

In an address to the International Congress of Mathematicians in 1900, David Hilbert posed a list of 23 problems he considered important for the future development of mathematics. The fifth of those problems concerned Lie groups. The problem stated by Hilbert was, "How far [is] Lie's concept of continuous groups of transformations…

approachable in our investigations without the assumption of the differentiability of the functions?" Exactly what Hilbert meant is open to debate. The most straightforward reading of this question would be to say, in modern language, that it asks whether a topological group whose underlying topological space is a manifold is in fact isomorphic as a topological group to a Lie group.

Hilbert's fifth problem, as I have formulated it here, was solved by Andrew Gleason [5]. In 1953, Hidehiko Yamabe proved a related and stronger result, which is a solution to a slightly different interpretation of the problem. Yamabe's result characterizes Lie groups among all locally compact groups. He showed that a locally compact group G is a projective limit of a sequence of Lie groups, and if G "has no small subgroups," then G is a Lie group.

Let's return to the first interpretation of Hilbert's problem, assuming that the underlying topological space of a topological group G is a manifold. Of course, G acts as a group of symmetries of its underlying space so that G is a group of symmetries of a manifold. Guided by this, and before all the work cited above on Hilbert's fifth problem was done, Smith generalized the problem by asking:

> *Conjecture 1. (Hilbert-Smith Conjecture). Let G be a locally compact topological group that is a group of symmetries of a (finite-dimensional) manifold. Then is G isomorphic as a topological group to a Lie group?*

Local compactness is necessary: For any manifold M of positive dimension the diffeomorphism group of M is infinite-dimensional, and hence is not locally compact and a *fortiori* not a Lie group. Because of Yamabe's results, it follows that to decide the Hilbert-Smith Conjecture it suffices to decide whether there is a prime p for which the additive group of p-adic integers can occur as the group of symmetries of a manifold. This problem remains open today.

Smith Theory

Most of Smith's work concentrated on the case of finite-order symmetries of spaces, usually of prime order p, and focused on the homological properties of the spaces and the subspaces fixed pointwise under the symmetry. In addition to reasonable local properties, the space was assumed to have the mod p homology of either Euclidean space of dimension n or the sphere of dimension n. In two articles, titled "Transformations of finite period" (1938) and "Transformations of finite period. II" (1939), Smith established fundamental results in this context—results that now carry the name "Smith

Theory." Smith began by asking himself, "To what extent does a finite-order symmetry of the sphere resemble an orthogonal transformation? For example, is its fixed point set a subsphere?" Brouwer [2] and Kérékjártó [6] had proved this for the 2-sphere, but the question was totally open in higher dimensions. As Smith said, "For $n > 2$, the difficulties that make a similar result seem unattainable at the present time are well-known to topologists. But if it is necessary to abandon the idea of determining completely the structure of periodic transformations it may nevertheless be of interest to study such of their properties as can be described in terms of homology theory." He showed the following:

> *Theorem 1. Fix a prime p. Suppose that X is a finite dimensional, compact Hausdorff space. Suppose that $\tau : X \to X$ is a symmetry with $\tau^p = Id_X$. If the mod p (Čech) homology of X is the same as that of an n-sphere, then the fixed point set of τ has the mod p (Čech) homology of a sphere.*

In "A theorem on fixed points for periodic transformations"(1934), Smith considered the natural extension of these results to Euclidean space. He showed:

> *Theorem 2. Fix a prime p. Suppose that X is a finite-dimensional locally compact Hausdorff space, suppose that the mod p (Čech) homology of X is the same as that of a point, and suppose that τ is a continuous symmetry of X of order p. Then the fixed point set of τ has the mod p (Čech) homology of a point.*

In both theorems, if X is a smooth manifold and τ preserves the smooth structure, then one can replace "Čech homology" with "singular homology."

There is also a local version of the first result: if X is n-dimensional, locally contractible, and has the local mod p homology of Euclidean n-space at every point, then the fixed point set F has the local mod p homology of a Euclidean space at every point and the dimension of the model Euclidean space is the same at every point of F.

> *Corollary 1. Any symmetry of Euclidean space of finite prime-power order has a fixed point.*

These two theorems constitute what is known as "Smith Theory."

Smith asked whether it was true more generally that symmetries of finite, but not prime-power, order of Euclidean space have a fixed point. This turns out not to be true. Pierre Conner and E. E. Floyd [3] showed that for distinct primes p and q, there is a smooth action of the cyclic group of order *pq* on Euclidean space without a fixed point. For example, there is such an action of the cyclic group of order 6. Even more, the Conner-Floyd construction can be modified such that for any finite simplicial complex *F* there is a smooth action of the cyclic group of order *pq* on a Euclidean space whose fixed point set is homotopy equivalent to *F*. Thus for any pair of distinct primes *p, q* there is a symmetry of order *p* of Euclidean space such that the mod *q* homology of its fixed point set is not the same as that of a point. Similarly, there are symmetries of a sphere of order *p* such that the mod *q* homology of its fixed point set is not the same as that of any sphere. This indicates that Smith had found the best statement of the extent to which the fixed point set of a finite-order symmetry resembles that of an orthogonal one: The symmetry should be assumed to be of prime-power order and one should consider homology with coefficients cyclic of order that prime.

Smith also considered more general groups of symmetries of the sphere. He asked what restrictions there are on the group-theoretic properties of a group *G* that is a finite group of symmetries of the sphere. This question has not been completely answered, though the result is now known for the case of the 3-sphere, using results of Grigori Perelman on the existence of geometric structures on 3-manifolds. Smith asked too: "If *G* is a finite group acting freely on a sphere, are all its *p*-primary subgroups (p an odd prime) cyclic? What about the case $p = 2$?" Smith himself answered these questions. In "Permutable periodic transformations" (1944), he showed that if *G* is a finite group acting freely on a sphere, then any Abelian subgroup of *G* is in fact cyclic. This implies that the *p*-primary subgroup of *G* is cyclic or for $p = 2$ either cyclic or a generalized quaternion group. Groups with this property are exactly the groups with periodic cohomology. But not all of them act freely on a sphere (see [7]).

Smith theory is properly viewed as the beginning of the equivariant cohomology theory of spaces with group actions, and in particular localization theorems relating cohomological invariants of the ambient space to those of the fixed points of a prime-power order symmetry. Equivariant cohomology developed into a full theory in the 1950s and '60s, primarily by Armand Borel, among many others. Localization theorems were developed by Sir Michael Atiyah, Raoul Bott, Graeme Segal, and Daniel Quillen, among others. Both of these areas remain of central importance throughout mathematics and theoretical physics today.

The Smith Conjecture

As Smith quickly realized, one case of Smith theory is especially interesting. When X is the 3-sphere and τ is orientation-preserving symmetry of prime order, it follows from Smith theory that the fixed point set of τ is a circle (or empty). This led Smith to ask, "If the fixed point set of a prime order, orientation-preserving symmetry of S^3 is nonempty, then is it an unknotted circle in S^3?" (See [4].)

What an obvious question! It turns out that without the assumption that τ is differentiable, the fixed point set may not be locally flat and hence not equivalent to the standard unknotted circle [8]. The statement that this is indeed the case when τ is differentiable became known as:

> *Conjecture 2. (The Smith Conjecture). If the fixed point set of a prime order, differentiable, orientation-preserving symmetry of S^3 is nonempty, then it is an unknotted circle in S^3.*

This conjecture remained open from its formulation in the 1940s until its resolution in 1978. (Friedhelm Waldhausen [9] had resolved the case $p = 2$ earlier.) The solution of the Smith Conjecture required major new ideas in 3-dimensional topology. From the purely topological arena, it required Waldhausen's theory of incompressible surfaces. From differential geometry came the equivariant version of Dehn's Lemma and the loop theorem due to William Meeks and Shing-Tung Yau. Above all, it required William Thurston's results about hyperbolic structures on 3-dimensional manifolds and, related to that, the work of Marc Cullen and Peter Shalen on groups acting on trees. (For more details, see [1] and the references therein.) All of these developments were far in the future and unimagined when Smith formulated the Smith Conjecture. The fact that it took so much sophisticated mathematics to resolve this elementary-sounding statement attests to its depth.

Smith's Method of Argument

Fix a prime p. Let X be a finite-dimensional locally compact metric space and τ a transformation of X of period p. Denote by F the fixed point set of τ. We consider the Čech chain complex of X with coefficients $\mathbb{F}_p$, the field with p elements. The transformation τ induces an action of C_p, the cyclic group of order p, on this complex, and hence the Čech complex of X is a chain complex of modules over the group ring $\mathbb{F}_p[C_p]$. Smith considers two elements, $\rho_0 = \sum_{i=0}^{p-1} \tau^i$ and $\rho_1 = 1 - \tau$, in this group ring. The symbol ρ stands for either ρ_0 or ρ_1 and if ρ is given in some discussion, then $\bar{\rho}$ is the other element.

The Smith homology group $H_r^\rho(X, F)$ is defined as follows: Consider the group relative cycles Z_r in (X, F) of the form ρW_r modulo F. We divide out by the group of boundaries of chains of degree $r + 1$ of the same type—i.e., we divide out by all relative cycles of the form $\partial\rho W_{k+1}$. Smith defines a descent process from $H_k^\rho(X, F)$ to $H_{k-1}^{\bar{\rho}}(X, F)$ as follows: Given a cycle Z_k modulo F of type ρ, $Z_k = \rho W_k$, the cycle $Z_{k-1} = \partial W_k$ is of type $\bar{\rho}$. Smith shows that this determines a well-defined map, the descent map, from $H_k^\rho (X, F) \to H_{k-1}^{\bar{\rho}} (X, F)$. Under the assumption that X is $\mathbb{F}_p$ an homolgy sphere of dimension n, Smith shows that if cycle modulo F of type ρ, say $Z_r = \rho W_r$ is nontrivial in Smith homology but ∂W_r vanishes in Smith homology, then W_r is a relative cycle in (X, F) representing a nontrivial class in $H_r (X, F)$.

Furthermore, one can reverse this process under one assumption. If Z_{k-1} is a cycle modulo F of type $\bar{\rho}$, then Z_{k-1} is an absolute cycle in X. If the class of Z_{k-1} is trivial in $H_{k-1} (X)$, i.e., if $Z_{k-1} = \partial W_k$ for some chain W_k then there is a cycle of Z_k of type ρ (namely ρW_k) representing a class in $H_k^\rho (X, F)$ whose descendant is $[Z_{k-1}]$. Now suppose that X has the homology of an n-sphere with $\mathbb{F}_p$ coefficients. Under this assumption, Smith showed that there is only one maximal nontrivial descent sequence in Smith homology. The initial (top) element of this sequence is the fundamental class of X and its bottom class is represented by ρW, where ∂W is a nontrivial homology class in F. In addition, F has no other reduced homology. Similarly, if X is acyclic with $\mathbb{F}_p$ coefficients, then there are no descent sequences and the reduced homology of F is trivial, meaning that F is acyclic with $\mathbb{F}_p$ coefficients.

In the case when X is a smooth manifold and τ is smooth, or in the case when X is a finite simplicial complex and τ is a simplicial map, we can take a more modern point of view of this construction. In either case there is a finitely generated chain complex $C_*(X)$ of modules over the group ring $\mathbb{F}_p[C_p]$, whose homology is identified with that of X (with the induced action of C_p being trivial on homology). There is a subchain complex $C_*(F)$ on which the C_p-action is trivial computing the homology of F. All chain groups of the quotient chain complex $C_*(X)/C_*(F)$ are free modules over $\mathbb{F}_p[C_p]$.

Let $C_*^{\rho_0}(X)$ and $C_*^{\rho_1}(X)$ be the subchain complexes of $C_*(X)$ given by $\rho_0 C_*(X)$ and $\rho_1 C_*(X)$, respectively. These complexes compute the Smith homology groups $H_*^{\rho_0}(X, F)$ and, $H_*^{\rho_1}(X, F)$ which follow from their homological properties. For consider the two exact sequences of complexes

$$0 \to C_*^{\rho_0}(X) \oplus C_*(F) \to C_*(X) \to C_*^{\rho_1}(X) \to 0$$

$$0 \to C_*^{\rho_1}(X) \oplus C_*(F) \to C_*(X) \to C_*^{\rho_0}(X) \to 0$$

Smith's results follow from the long exact homology sequences associated with these short exact sequences of chain complexes. In particular, Smith's descent process is induced from the connecting homomorphisms of the homology of these sequences.

This modern description makes the result clearer, but it does not do justice to what Smith accomplished. Smith was working with Čech chains rather than finite-dimensional chain complexes. He had many technical difficulties with open coverings and their refinements. Also, tools such as the algebraic theory of resolutions and the quasi-isomorphisms of chain complexes had not yet been developed. In spite of all this, Smith managed to discover and express fairly clearly the ingredients of homological algebra that he needed for his study. Not only had he focused on the correct statement, he also gave a proof in a technically difficult context that exposed essential aspects of the needed but yet-to-be-developed homological algebra. Finally, Smith's argument gives a strictly stronger result than the smooth or simplicial results, in that there are actions whose fixed point sets are not topologically equivalent to fixed point sets of smooth or simplicial actions.

The Final Word

There was a conference at Columbia in 1979 that brought together all of the main actors to explain their work and its application to resolving the Smith Conjecture. Smith sat front-row center for the entire conference and, ever the gentleman, paid rapt attention as the sophisticated geometry, algebra, and topology needed to resolve his conjecture was presented. At the end of the conference, he was asked for his thoughts. He replied, "I will try to make my next conjecture easier."

Sadly, he died a year later on June 13, 1980.

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