NATIONAL ACADEMY OF SCIENCES

MARK KAC

1914—1984

A Biographical Memoir by

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Biographical Memoir

MARK KAC

August 16, 1914–October 25, 1984

BY H. P. MCKEAN

POLAND. Mark Kac was born "to the sound of the guns of August on the 16th day of that month, 1914," in the town of Krzemieniec—then in Russia, later in Poland, now in the Soviet Ukraine (1985,1, p. 6). In this connection Kac liked to quote Hugo Steinhaus, who, when asked if he had crossed the border replied, "No, but the border crossed me."

In the early days of the century Krzemieniec was a predominantly Jewish town surrounded by a Polish society generally hostile to Jews. Kac's mother's family had been merchants in the town for three centuries or more. His father was a highly educated person of Galician background, a teacher by profession, holding degrees in philosophy from Leipzig, and in history and philosophy from Moscow.

As a boy Kac was educated at home and at the Lycée of Krzemieniec, a well-known Polish school of the day. At home he studied geometry with his father and discovered a new derivation of Cardano's formula for the solution of the cubic—a first bite of the mathematical bug that cost Kac *père* five Polish zlotys in prize money. At school, he obtained a splendid general education in science, literature, and history. He was grateful to his early teachers to the end of his life.

In 1931 when he was seventeen, he entered the John

Casimir University of Lwów, where he obtained the degrees M. Phil. in 1935 and Ph.D. in 1937.

This was a period of awakening in Polish science. Marian Smoluchowski had spurred a new interest in physics, and mathematics was developing rapidly: in Warsaw, under Wacław Sierpinski, and in Lwów, under Hugo Steinhaus. In his autobiography (1985,1, p. 29), Kac called this renaissance "wonderful." Most wonderful for him was the chance to study with Steinhaus, a mathematician of perfect taste, wide culture, and wit; his adored teacher who became his true friend and introduced him to the then undigested subject of probability. Kac would devote most of his scientific life to this field and to its cousin, statistical mechanics, beginning with a series of papers prepared jointly with Steinhaus on statistical independence (1936,1–4; and 1937,1–2).

Kac's student days saw Hitler's rise and consolidation of power, and he began to think of quitting Poland. In 1938 the opportunity presented itself in the form of a Polish fellowship to Johns Hopkins in Baltimore. Kac was twenty-four. He left behind his whole family, most of whom perished in Krzemieniec in the mass executions of 1942–43. Years later he returned, not to Krzemieniec but to nearby Kiev. I remember him rapt, sniffing about him and saying he had not smelled such autumn air since he was a boy. On this trip he met with a surviving female cousin who asked him, at parting, "Would you like to know how it was in Krzemieniec?" then added, "No. It is better if you don't know" (1985,1, p. 106).

These cruel memories and their attendant regrets surely stood behind Kac's devotion to the plight of Soviet *refusniks* and others in like distress. His own life adds poignancy to his selection of the following quote from his father's hero, Solomon Maimon: "In search of truth I left my people, my country and my family. It is not therefore to be assumed that I shall forsake the truth for any lesser motives" (1985,1, p. 9).

AMERICA

Kac came to Baltimore in 1938 and wrote of his reaction to his new-found land:

"I find it difficult . . . to convey the feeling of decompression, of freedom, of being caught in the sweep of unimagined and unimaginable grandeur. It was life on a different scale with more of everything—more air to breathe, more things to see, more people to know. The friendliness and warmth from all sides, the ease and naturalness of social contacts. The contrast to Poland . . . defied description." (1985,1, p. 85)

After spending 1938–39 in Baltimore, Kac moved to Ithaca, where he would remain until 1961. Cornell was at that time a fine place for probability: Kai-Lai Chung, Feller, Hunt, and occasionally the peripatetic Paul Erdös formed, with Kac, a talented and productive group. His mathematics bloomed there. He also courted and married Katherine Mayberry, shortly finding himself the father of a family. So began, as he said, the healing of the past.

From 1943 to 1947 Kac was associated off and on with the Radiation Lab at MIT, where he met and began to collaborate with George Uhlenbeck. This was an important event for him. It reawakened his interest in statistical mechanics and was a decisive factor in his moving to be with Uhlenbeck at The Rockefeller University in 1962. There Detlev Bronk, with his inimitable enthusiasm, was trying to build up a small, top-flight school. While this ideal was not fully realized either then or afterwards, it afforded Kac the opportunity to immerse himself in the statistical mechanics of phase transitions in the company of Ted Berlin and Uhlenbeck, among others. Retiring in 1981, Kac moved to the University of Southern California, where he stayed until his death on October 25, 1984, at the age of seventy. He is survived by his wife Kitty, his son Michael, his daughter Deborah, and his grandchildren.

MATHEMATICAL WORK

Independence and the Normal Law

In the beginning was the notion of statistical independence to which Steinhaus introduced Kac. The basic idea is that the probability of the joint occurrence of independent events should be the product of their individual probabilities, as in $1/2 \times 1/2 = 1/4$ for a run of two heads in the tossing of an honest coin. The most famous consequence of this type of independence is the fact that, if #(n) is the number of heads in n tosses of such an honest coin, then the *normal law of errors* holds:

$$P\left[a \leq \frac{\#(n) - n/2}{(1/2)\sqrt{n}} \leq b\right] \simeq \int_a^b (2\pi)^{-1/2} e^{-x^2/2}\, dx$$

in which P signifies the probability of the event indicated between the brackets, the subtracted n/2 is the mean of #(n), and the approximation to the right-handed integral improves indefinitely as n gets large. The fact goes back to A. de Moivre (1667–1754) and was extended to a vague but much more inclusive statement by Gauss and Laplace. It was put on a better technical footing by P. L. Čebyšev (1821–1890) and A. A. Markov (1856–1922), but as Poincaré complained, "*Tout le monde y croit (la loi des erreurs) parce que les mathématiciens s'imaginent que c'est un fait d'observation et les observateurs que c'est un théorème de mathématiques.*" The missing ingredient, supplied by Steinhaus, was an unambiguous concept of independence. But that was only the start. All his life Kac delighted in extending the sway of the normal law over new and unforseen domains. I mention two instances:

Let $\omega_1,\ldots,\omega_n$ be n ($\geq$ 2) independent frequencies, meaning that no integral combination of them vanishes. Then

$$T^{-1} \text{ measure} \left[0 \leq t \leq T: a \leq \frac{\sin \omega_1 t + \ldots + \sin \omega_n t}{(1/2)\sqrt{n}} \leq b\right]$$
$$\simeq \int_a^b (2\pi)^{-1/2} e^{-x^2/2} dx$$

for large T and n, in which *measure* signifies the sum of the lengths of the several subintervals on which the indicated inequality takes place. In short, sinusoids of independent frequencies behave as if they were statistically independent, though strictly speaking they are not (1937,2; 1943,2).

On another occasion, Kac looked to a vastly different domain: Let d(n) be the number of distinct prime divisors of the whole number n = 1,2,3, ... Then for large N,

$$N^{-1}\#\left[n \leq N: a \leq \frac{d(n) - lg_2 n}{2\sqrt{lg_2 n}} \leq b\right] \simeq \int_a^b (2\pi)^{-1/2} e^{-x^2/2} dx$$

in which # denotes the number of integers having the property indicated in the brackets and $lg_2 n$ is the iterated logarithm $lg(lgn)$. In short, there is some kind of statistical independence in number theory, too. Kac made this beautiful discovery jointly with P. Erdös (1940,4).

These and other examples of statistical independence are explained in Kac's delightful Carus Monograph, *Independence in Probability, Analysis and Number Theory* (1951,1).

Brownian Motion and Integration in Function Space

The Brownian motion, typified by the incessant movement of dust motes in a beam of sunlight, was first discussed from a physical standpoint by M. Smoluchowski and A. Einstein (1905). N. Wiener later put the discussion on a solid mathematical footing. Kac was introduced to both developments during his association with MIT from 1943 to 1947.

Now the statistical law of the Brownian motion is *normal*: if x(t) is the displacement of the Brownian traveler in some fixed direction, then

$$P[a \leq x(t) \leq b] = \int_a^b (2\pi t)^{-1/2} e^{-x^2/2t} dx.$$

The fact is that Brownian motion is nothing but an approximation to honest coin-tossing:

$$x(t) \simeq \frac{1}{\sqrt{N}} \times [+ \text{ (the number of heads in T tosses)} - \text{(the number of tails)}],$$

in which T is the whole number nearest to tN and N is large. The normal law for coin-tossing cited before is the simplest version of this approximation. Kac, with the help of Uhlenbeck, perceived the general principal at work, of which the following is a pretty instance: Let p(n) be the number of times that heads outnumber tails in n tosses of an honest coin. Then the *arcsine law* holds:

$$P[n^{-1}p(n) \leq c] \simeq \frac{1}{\pi}\int_0^c \frac{dx}{\sqrt{x(1-x)}} = \frac{2}{\pi} \text{ arcsine } \sqrt{c},$$

the right-hand side being *precisely* the probability that the Brownian path x(t): $0 \leq t \leq 1$, starting at $x(0) = 0$, spends a total time, $T \leq c$, to the right of the origin (1947,2).

Kac's next application of Brownian motion was suggested in a quantum-mechanical form by R. Feynman. It has to do with the so-called elementary solution e(t,x,y) of the Schrödinger equation:

$$\sqrt{-1}\ \partial\psi/\partial t = \partial^2\psi/\partial x^2 - V(x)\psi.$$

The formula states that, *with the left-hand imaginary unit* $\sqrt{-1}$ *removed*:

$$e(t,x,y) = E_{xy}\, e^{-\int_0^t V[x(t')]dt'} \times (4\pi t)^{-1/2}\, e^{-(x-y)^2/4t}$$

in which the final factor is the *free* elementary solution (for $V = 0$) and Exy is the Brownian mean taken over the class of paths starting at $x(0) = x$ and ending at $x(t) = y$. This is not really as explicit as it looks, as the mean is not readily expressible in closed form for any but the simplest cases, but it does exhibit just how e depends upon V in a transparent way. It can be used very effectively, as Kac illustrated by a beautiful derivation of the WKB approximation of classical quantum mechanics (1946,3).

I will describe one more application of Brownian motion contained in Kac's Chauvenet Prize paper, *Can One Hear the Shape of a Drum*? (1976,1). The story goes back to H. Weyl's proof of a conjecture of H. A. Lorentz. Let D be a plane region bounded by one or more nice curves, holes being permitted, and let ω_1, ω_2, etc., be its fundamental tones, i.e., let $-\omega_1^2$, ω_2^2, etc., be the eigenvalues of Laplace's operator $\Delta = \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2$ acting upon smooth functions that vanish at the boundary of D. Then Lorentz conjectured and Weyl proved:

$$\#[n: \omega_n \leq \omega] \simeq \pi^{-1}\omega^2 \times \text{the area of D}$$

for large ω. Kac found a remarkably simple proof of this fact based upon the self-evident principle that the Brownian traveller, starting inside D, does not feel the boundary of D until it gets there. He also speculated as to whether you could deduce the shape of D (up to rigid motions) if you could "hear" all of its fundamental tones and showed that, indeed, you can hear the length of the boundary, and the number of holes if any. The full question is still open.

Statistical Mechanics

As noted before, Kac's interest in this subject had been reawakened by Uhlenbeck at MIT. A famous conundrum of the field was the superficial incompatibility of the (obvious) *irreversibility* of natural processes and the *reversibility* of the underlying molecular mechanics. Boltzmann struggled continually with the problem, best epitomized by Uhlenbeck's teachers, P. and T. Ehrenfest, in what they called the "dog-flea" model. Kac's debut in statistical mechanics was to provide its complete solution, put forth in his second Chauvenet Prize paper (1947,4).

Next, Kac took up Boltzmann's equation describing the development, in time, of the distribution of velocities in a dilute gas of like molecules subject to streaming and to collisions (in pairs). I think this work was not wholly successful, but it did prompt Kac to produce a stimulating study of Boltzmann's idea of "molecular chaos" (*Stosszahlansatz*) and a typically elegant, Kac-type "caricature" of the Boltzmann equation itself.

I pass on to the eminently successful papers on phase transitions. The basic question which Maxwell and Gibbs answered in principle is this: How does steam know it should be water if the pressure is high or the temperature is low, and how does that come out of the molecular model? There are as many variants of the question as there are substances. A famous one is the Ising model of a ferromagnet, brilliantly solved by L. Onsager in the two-dimensional case. Kac and J. C. Ward found a different and much simpler derivation (1952,2). The related "spherical model" invented by Kac was solved by T. Berlin (1952,1).

But to my mind, Kac's most inspiring work in this line is contained in the three papers written jointly with P. C. Hemmer and Uhlenbeck (1963,1–3), in which they related the phase transition of a one-dimensional model of a gas to the splitting of the lowest eigenvalue of an allied integral equation and derived, for that model, the (previously *ad hoc*) van der Waal's equation of state, Maxwell's rule of equal areas included—a real *tour de force*.

PERSONAL APPRECIATION

I am sure I speak for all of Kac's friends when I remember him for his wit, his personal kindness, and his scientific style. One summer when I was quite young and at loose ends, I went to MIT to study mathematics, not really knowing what that was. I had the luck to have as my instructor one M. Kac and was enchanted not only by the content of the lectures but by the person of the lecturer. I had never seen mathematics like that nor anybody who could impart such (to me) difficult material with so much charm.

As I understood more fully later, his attitude toward the subject was in itself special. Kac was fond of Poincaré's distinction between God-given and man-made problems. He was particularly skillful at pruning away superfluous details from problems he considered to be of the first kind, leaving the question in its simplest interesting form. He mistrusted as insufficiently digested anything that required fancy technical machinery—to the extent that he would sometimes insist on clumsy but elementary methods. I used to kid him that he had made a career of noting with mock surprise that ex = 1 + x + $x^2/2$ + etc. when the whole thing could have been done without expanding anything. But he did wonders with these sometimes awkward tools. Indeed, he loved computation (*Desperazionsmatematik* included) and was a prodigious, if secret, calculator all his life.

I cannot close this section without a Kac story to illustrate his wit and kindness. Such stories are innumerable, but I reproduce here a favorite Kac himself recorded in his autobiography:

"The candidate [at an oral examination] was not terribly good—in mathematics at least. After he had failed a couple of questions, I asked him a really simple one . . . to describe the behavior of the function 1/z in the complex plane. 'The function is analytic, sir, except at z = 0, where it has a singularity,' he answered, and it was perfectly correct. 'What is the singularity called?' I continued. The student stopped in his tracks. 'Look at me,' I said. 'What am I?' His face lit up. 'A simple Pole, sir,' which was the correct answer." (1985,1, p. 126)

HONORS, PRIZES, AND SERVICE

1950	Chauvenet Prize, Mathematical Association of America
1959	American Academy of Sciences
1963	Lorentz Visiting Professor, Leiden
1965	National Academy of Sciences
1965–1966	Vice President, American Mathematical Society
1966–1967	Chairman, Division of Mathematical Sciences, National Research Council
1968	Chauvenet Prize, Mathematical Association of America
1968	Nordita Visitor, Trondheim
1969	Visiting Fellow, Brasenose College, Oxford
1969	American Philosophical Society
1969	Royal Norwegian Academy of Sciences
1971	Solvay Lecturer, Brussels
1976	Alfred Jurzykowski Award
1978	G. D. Birkhoff Prize, American Mathematical Society
1980	Kramers Professor, Utrecht
1980	Fermi Lecturer, Scuola Normale, Pisa

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