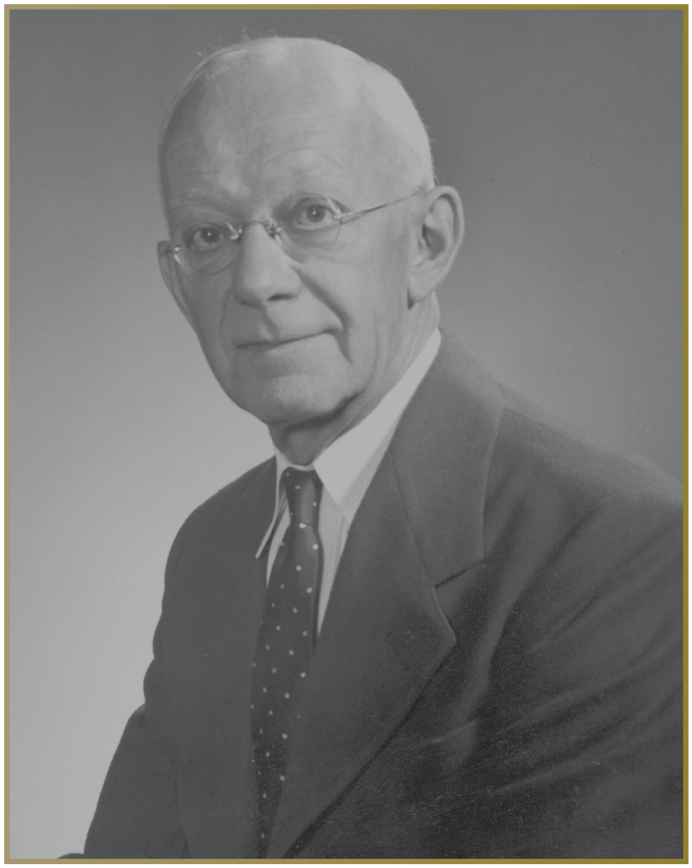
BIOGRAPHICAL MEMOIRS

Arthur Coble

November 3, 1878–December 8, 1966
Elected to the NAS, 1924

A Biographical Memoir by János Kollár



Arthur Byron Coble was a noted mathematician whose research focused on algebraic geometry and group theory. He was also an academic who taught for more than forty years. In addition to his academic career, he served the mathematics community in various positions, including the presidency of the American Mathematical Society (AMS) and as editor of the *Transactions of the American Mathematical Society*. Several geometric terms were named in his honor, including the Coble surface and the Coble hypersurfaces.[1,2,3]

Life and Career

Coble was born November 3, 1878, in Dauphin County, Pennsylvania, near Harrisburg. He graduated from Pennsylvania College (now Gettysburg College) in 1897. After a year of teaching, he studied at the Johns Hopkins University with Frank Morley, earning a Ph.D. in 1902 with a dissertation entitled *The Quartic Curve as Related to Conics*. After a year as an instructor at the University of Missouri, he returned to Johns Hopkins as a research assistant in 1903, with funding from the Carnegie Institution. He was promoted to research associate and later to associate professor. In 1904, he visited Greifswald University and the University of Bonn in Germany, also with support from the Carnegie Institution.

In 1918, he accepted a professorship at the University of Illinois (now the University of Illinois Urbana-Champaign) and spent the rest of his career there except for visiting positions at the University of Chicago (1919) and the Johns Hopkins University (1927–1928). In 1933, Coble became head of the math department and served in that position until his retirement in 1947.

During his academic career, Coble supervised twenty-seven students, among them seven women, starting with Bessie Miller (Johns Hopkins, 1914) and ending with Janie Lapsley Bell (University of Illinois, 1943). He was among the leading advisors for women doctoral students in mathematics before 1940. Coble's students did not seem to have continued his work in algebraic geometry. He has more than 200 academic descendants, about half of whom wrote theses in applied mathematics and automata theory, and the other half in mathematics education.[4,5,6]

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Coble was very active in the AMS, served on the governing council (1911–1914), as vice president (1917–1920), as chair of the Chicago section (1922), and as president (1933–1934). He was editor of the *Transactions of the American Mathematical Society* (1920–1925), *Proceedings of the American Mathematical Society* (1933–1934), and the *Duke Mathematical Journal* (1936–1938). He served several times on the National Research Council and on investigating committees of the American Association of University Professors. He was elected to the National Academy of Sciences in 1924, delivered the AMS Colloquium lectures in 1928, and received an honorary degree from Gettysburg College in 1932.

After his retirement, Coble moved back to Dauphin County, Pennsylvania, and died in Harrisburg on December 8, 1966.

Mathematical Works

Coble published numerous research papers on his work. His most important contribution is the book *Algebraic Geometry and Theta Functions*.[7] A detailed review by Oscar Zariski notes that "thanks to its rich geometric content and originality of treatment, Coble's work is a really important contribution to the theory and application of the θ-functions." Zariski also notes that Coble's book is for "the competent readers."[8]

A more precise title would have been "Cremona transformations and theta functions," two topics that had been very actively investigated at that time. Besides giving a broad overview, the book also contains a good description of Coble's research up to that time, with many new results added. Coble wrote fewer papers after this book, so it gives an almost complete picture of his research.

Although not as influential as some other volumes in that series around the same time (for example by Solomon Lefschetz, Marston Morse, or Marshall Stone), Coble's book was reprinted in 1962 and again in 1980 and still earns an average of four citations each year.

The books by Yuri I. Manin (*Cubic forms: algebra, geometry, arithmetic, Nauka, Moscow*, 1972) and David Mumford (*Tata lectures on theta. I-III*. Progress in Mathematics, vols. 28, 43, 97. Birkhäuser, Boston, MA, 1983–1991) both cite Coble in passing, but neither gives justice to the wealth of material in his book. This was remedied by Igor Dolgachev and David Ortland (*Points Sets in Projective Spaces and Theta Functions*, Astérisque, vol. 165, 1988).

Coble's book appeared at a transitional period of algebraic geometry. Much of the earlier work was devoted to the geometry of subvarieties of projective spaces, with emphasis on especially nice and interesting examples in low dimensions. Around that time, Bartel van der Waerden, André Weil, and Zariski were reworking algebraic geometry to focus on general theorems and free it from the constraints of the projective space.

Coble also moved beyond the linearity imposed by projective geometry by looking at properties invariant not by the linear automorphism group PGL_{n+1}, but by the birational automorphism group Cr_n, called the Cremona group in dimension n. In hindsight, although Coble made a significant step in the right direction, it was not decisive enough.

One of Coble's major discoveries relates the plane Cremona group Cr_2 to the sequence of Coxeter-Weyl groups of the sequence of lattices

$$A_1 + A_2, A_4, D_5, E_6, E_7, E_8, E_9, E_{10}, \dots$$

(Kunihiko Kodaira denotes E_9 by $\tilde{E}_8$, and, especially for $m \geq 9$, these are also frequently denoted by $T_{3,2,m-3}$.)

Using modern terminology, let S_m be a surface obtained by blowing up m points in P^2 (in general position) and let $E_m \subseteq H_2(S_m,Z)$ be the orthogonal complement of the canonical class. Poincaré duality gives a quadratic form, which is indefinite for $m \geq 10$. These are exactly the lattices mentioned above.

Let $W(E_m)$ denote the Coxeter-Weyl group of E_m. Coble observes that there are some natural representations of $W(E_m)$ on the configuration space of m points in the plane. For $m \leq 8$, these give rise to the construction of the moduli space of Del Pezzo surfaces of degree $9 - m$. The case $m = 6$ corresponds to cubic surfaces; these were much studied earlier. Coble also gives detailed information about the $m = 7,8$ cases, the latter one being especially complicated.

One of Coble's major discoveries is that if we blow up the 10 singular points of a rational sextic curve, then the automorphism group is a finite index subgroup of the Coxeter-Weyl group $W(E_{10})$. A difficult theorem of Serge Cantat and Dolgachev shows that these—now called Coble surfaces—are the only ones with this property in characteristic 0[9].

Coble surfaces also play an important role in the theory of Enriques Surfaces. Dolgachev and Shigeyuki Kondō devote an entire chapter of their monograph *Enriques Surfaces II* to their detailed study.[10]

Coble also investigated a remarkable partial inverse of the map $W(E_m) \rightarrow Cr_2$. The sub-groupoid of Cr_2 consisting of Cremona transformations with $\leq m$ base points maps to $W(E_m)$. (As Coble notes, there is an issue here with ordered versus unordered sets of base points, so this needs to be done more carefully.) In some vague sense, this suggests that Cr_2 might act on some hyperbolic space.

It turns out that Cr_2 cannot act on a finite dimensional space. However, if we let the number of blown-up points go to infinity, more precisely, if we blow up all the points, then we get a representation of Cr_2 on an infinite dimensional hyperbolic space. This surprising result, made by Cantat, leads to a series of deep results on Cr_2.[11] It is unlikely that Coble actually foresaw this development, but his work was the first to establish some connection between Cr_2 and hyperbolic groups.

For theta functions, from Coble's point of view, a key earlier result is a theorem of Carl Johannes Thomae (1870) that, in modern language, describes the coordinate ring of the geometric invariant theory quotient of the configuration space of n points in P^1 using theta functions. Another result—going back at least to Julius Plücker (1839)—is given by the bitangents of plane quartic curves (a topic of surprisingly many papers). Bitangents are linear objects, but they are also the odd theta characteristics of genus 3 curves and give the generators of the Kleiman-Mori cone of degree 2 Del Pezzo surfaces. So a property in classical linear geometry turns out to be equivalent to abstract properties of curves and surfaces. Coble's AMS Colloquium lecture was devoted to the genus 4 case, where the odd theta characteristics correspond to tritangent planes of the degree 6 canonical image of the curve in P^3. Much of Coble's book can be viewed as a search for far-reaching generalizations of these examples.

Dolgachev points out that, for a historian of mathematics, Coble's book is frustratingly haphazard with citations. The preface clearly states which sections contain new material, but facts frequently appear in the other sections without citations and without clear indications of how the proof should go. The basics of the Cremona representations of the Weyl groups were known to Seligmann Kantor, and the theory of "association" (now called Gale transform) can be traced back to Guido Castelnuovo. It is also not clear where the results on the Castelnuovo-Igusa quartic threefold come from. (As the current names of these objects indicate, Coble was not the last one to rediscover old theorems and examples.)

ACKNOWLEDGMENTS

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