Walter Feit

1930–2004

BIOGRAPHICAL *Memoirs*

A Biographical Memoir by
Leonard L. Scott
and Ronald Solomon

NATIONAL ACADEMY OF SCIENCES

WALTER **FEIT**

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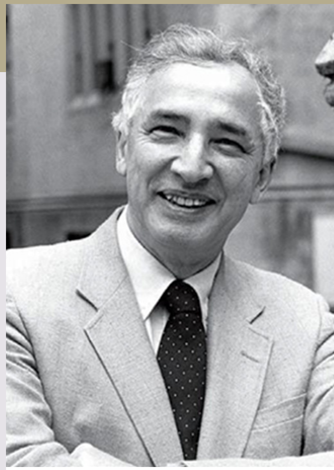

By Leonard L. Scott and Ronald Solomon

The greatest single paper in the history of finite group theory was the paper "Solvability of Groups of Odd Order"[1] by Walter Feit and John G. Thompson, published in 1963 in the *Pacific Journal of Mathematics* and forever after known as the Odd Order Paper. For this paper, Feit and Thompson were honored with the 1965 Cole Prize in Algebra of the American Mathematical Society. But even this fine honor fails to capture the revolutionary significance of this paper. To paraphrase Richard Brauer, before the Odd Order Paper, no one had any idea how to study finite simple groups. After the Odd Order Paper, there was a wealth of ideas and methods sufficient to enable the classification of finite simple groups, completed 40 years later.

Walter Feit was born on October 26, 1930, the 81st birthday of Ferdinand G. Frobenius, whose work was to have a profound impact on Feit's own research. He was born in Vienna, Austria, to shopkeepers Paul and Esther Feit. Walter's childhood was a tragic time for the people of Europe, especially the Jewish people. At the age of eight in August 1939, Walter's parents sent him on a Kindertransport to London. He never saw his parents again.

Shortly after his arrival in London, the evacuation of children from London began, and he was eventually moved to Oxford where, in 1943, he won a scholarship to an Oxford technical high school. At this time, he became "passionately interested in mathematics." After the war, he joined relatives in Florida and, in 1947, entered the University of Chicago, graduating in 1951 with bachelor's and master's degrees. Probably during this period, Feit came upon a copy of William Burnside's *Theory of Finite Groups* in the Eckhart Library and studied it, including the famous Note N, in which Burnside conjectures that all non-abelian finite simple groups have order divisible by 2. Feit mentioned that the library copy contained a handwritten marginal comment by Leonard E. Dickson, "also by 3." (Dickson was wrong, as later shown by Michio Suzuki.)

Upon graduation, Feit enrolled at the University of Michigan to study with Richard Brauer, who had joined the faculty there in 1948. Brauer was a mathematical grandchild of Frobenius and the leading expert in group representation theory. In January 1952, Michio Suzuki came to the United States and spent the summer of 1952 at the University of Michigan with Brauer and Feit. This was a momentous confluence of minds. In autumn 1952, Brauer moved to Harvard University, and Feit remained at Michigan but continued under Brauer's supervision. One significant portion of Feit's doctoral thesis was a new proof of a celebrated Frobenius theorem using Brauer's characterization of characters. The theorem asserts that if G is a transitive permutation group that is not regular, but in which no non-identity element fixes two points, then G has a regular normal subgroup K. Such groups came to be known as Frobenius groups with Frobenius kernel K. The other part of Feit's thesis evolved into a joint paper with Brauer[2] bounding the number of characters in a p-block. The bound is $\frac{1}{4}p^{2d}+1$. Their result remains the best known, though it exceeds the conjectured upper bound.

Feit accepted an instructorship at Cornell University in Ithaca, New York, in 1953. His academic career was interrupted by military service but then resumed at Cornell. He married Sidnie Dresher in 1957. They had two children, Paul (b. 1959), a mathematician on the faculty of the University of Texas of the Permian Basin, and Alexandra (b. 1961), an artist now living in Haines, Alaska. Walter and Paul produced the joint paper "The K-Admissibility of $SL(2, 5)$."

Building on fundamental work of Jordan and Frobenius, Hans Zassenhaus initiated in 1936 the study of 2-transitive permutation groups with no regular normal subgroup in which every non-identity element fixes at most two points. Under additional hypotheses, he was able to show that such a group G has a subgroup isomorphic to $PSL(2, q)$ of index 1 or 2. Such groups came to be called Zassenhaus groups. The problem of classifying all Zassenhaus groups was pursued in the 1950s by Feit, Suzuki, and Noboru Ito. In a Zassenhaus group, a 1-point stabilizer, N, is a Frobenius group, and in his 1958 dissertation, John G. Thompson proved that a Frobenius kernel is necessarily a nilpotent group. In the same year, Feit announced the following result in the article "On Groups Which Contain Frobenius Groups as Subgroups:"[3]

Theorem 1. *Let G be a finite group with a T.I. subgroup M ($M \cap M^g = 1$ for all $g \in G - N_G(M)$) such that $N := N_G(M)$ is a Frobenius group. Let $m = |M|$, $h = |N : M|$, and let $k + 1$ be the number of irreducible characters of M. Suppose that*

(1) $h \neq m - 1$; *and*
(2) M *is not a non-abelian group with* $|M : M'| < 4h^2$.

Then the $\frac{k}{h} + 1$ characters of G induced by those of M give rise, in a natural way, to $\frac{k}{h}$ irreducible characters of G.

As noted in G. E. Wall's review of the theorem, "This in a sense refines the 'exceptional character' theory of Brauer and Suzuki." As an application, Feit proved the following theorem in "On a Class of Doubly Transitive Permutation Groups,"[4] which provided proofs for the results in Theorem 1:

Theorem 2. *Let G be a Zassenhaus group with M, m and h as in Theorem 1. Then $m = p^e$ for some prime p, and $|M : M'| < 4h^2$. Moreover, if $M' = 1$, then G has a subgroup G_0 of index at most 2 with $G_0 \cong PSL(2, p^e)$.*

Note that in Theorem 2, *M* is the Frobenius kernel of the Frobenius group $N = N_G(M)$, which is the stabilizer of a point in *G*. Thus, by Thompson's thesis, *M* is a nilpotent group, i.e., a product of groups of prime power order. Theorem 2 proves that in fact *M* is a *p*-group for some prime *p*, as well as bounding $|M : M'|$. Shortly thereafter Suzuki treated the case when $M' = 1$, thereby completing the classification of Zassenhaus groups and discovering in the process an infinite family of simple groups, *Sz*(*q*), all of order not divisible by 3. Here $q = 2^{2n+1}$, $n \geq 1$, $m = q^2$, $h = q - 1$, and $|M : M'| = q$. Suzuki's discovery was a surprise to Feit, who was biased (by Dickson's marginal comment?) to believe that all finite simple groups would have an order divisible by 3.

One main ingredient of Feit's new contribution to exceptional character theory was that it applied to the case of non-abelian Frobenius kernel M. Indeed, in a footnote in the 1960 article, Feit commented that Suzuki had informed him that he, Suzuki, and Zassenhaus had been able to prove one of Feit's main permutation group results on G when M was abelian.

To briefly explain why the abelian case is easier, note that all the (absolutely) irreducible characters ζ of M have the same degree 1 in that case. The difference $\zeta - \zeta'$ of any two of them restricts to 0 on the trivial subgroup, and the T. I. property (by a character calculation, or use of the Mackey decomposition theorem) then implies that the difference induces to a difference of two irreducible characters of G. The Brauer-Suzuki theory[4] then implies that, by making a choice of sign ϵ and a choice of character assignments $\zeta \to \chi, \zeta' \to \chi', \ldots$, that $\zeta - \zeta'$ induces to $\epsilon(\chi - \chi')$, $\zeta' - \zeta''$ induces to $\epsilon(\chi' - \chi''), \ldots$, etc. The ability to make this choice was later to be called "coherence" in a more general context (see below). Analogous choices can't always be made for all the irreducible characters of M in the non-abelian case, but Feit was able to prove a somewhat more restricted version when M was nilpotent. See Wall's review of Feit's 1959 article above, as well as the 1960 article and the relevant chapter in Feit's character theory book.[5] The last quoted result uses the newer terminology "coherence" in its statement and uses in its proof more general and elaborate results from the Odd Order Paper.[1] Coherence is simultaneously a simple but potentially difficult concept: Following Feit's ideas in his 1967 book, one starts with a set of possibly reducible characters of a subgroup N of a finite group G, setting $I(S)$ to be the set of all integer linear combinations of elements of S, and denoting by $I_0(S)$ the members of $I(S)$, which vanish at 1. A linear isometry τ from $I_0(S)$ to the integral character ring of G is said to be coherent if it can be extended to a linear isometry on all of $I(S)$. If τ is understood from context, then S is said to be coherent if τ is coherent.

It seems evident, from the linkage of Theorem 2[4] and Feit's additional analysis in his 1967 work[5] that his Zassenhaus group work helped inspire the coherence notion and its later, more difficult application in the Odd Order Paper. The notion itself was not enough, however, and more sophistication was needed, beyond constructing candidate isometries τ from induction using T.I. sets. Feit writes near the beginning of "Isometries," the last section of Chapter IV of his character theory book: "For some purposes the [T.I. set] assumption is too restrictive. In Feit-Thompson[1] such a map τ was constructed under weaker hypotheses."[5]

Feit goes on to describe a simplification and generalization of that work owing to Dade[6] and presents Dade's method in the rest of the chapter. Further work on coherence was later done by Sibley, some of it appearing in "Coherence in Finite Groups Containing a Frobenius Section."[7] A full exposition of the character theory needed in the Odd Order Paper, including the contributions of Dade and Sibley, was written by Peterfalvi.[8] Chapter 4 of *Characters of Finite Groups* remains a good introduction, both to the Odd Order paper character theory and to the context of Peterfalvi's much later treatment. Written ostensibly for students, it gives much insight into Feit's thinking and does treat a nontrivial case of the Odd Order Paper of historical interest, namely the CN theorem of Feit, Marshall Hall, and Thompson,[9] inspired by the work of Suzuki, which we now discuss.

In "The Nonexistence of a Certain Type of Simple Groups of Odd Order,"[10] Suzuki achieved the first significant result on groups of odd order since the work of William Burnside early in the twentieth century. He proved:

CA Theorem. *A finite group of odd order must be solvable if the centralizer of every non-identity element is abelian.*

The structure of the proof is the following: Let *G* be a minimal counterexample to the theorem. Thus, every proper subgroup of *G* is solvable. Using the centralizer hypothesis, Suzuki is able to prove that every maximal subgroup of *G* is a Frobenius group with abelian kernel. Using this, he is able to deduce detailed information about the ordinary characters of the group *G*, and this suffices to yield a contradiction. Recall that Thompson proved in his doctoral dissertation that the kernel of every Frobenius group is a nilpotent group. Marshall Hall, who had recently taken a faculty position at the California Institute of Technology, invited Feit and Thompson to visit with the goal of extending Suzuki's result. Walter and Sidnie's son Paul had just been born, and so Walter was unable to accept the invitation. Hall and Thompson succeeded in improving Suzuki's result to the following:

CN Theorem. *A finite group of odd order must be solvable if the centralizer of every non-identity element is nilpotent*.

The structure of the proof is again: Let *G* be a minimal counterexample to the theorem. The centralizer hypothesis makes it possible to prove that every maximal subgroup is a Frobenius group. Using this, it is possible to deduce detailed information about the ordinary characters of *G* and finally to derive a contradiction. Hall and Thompson sent a draft of their paper to Walter, who was able to greatly improve the character theoretic portion of the argument, even improving on Suzuki's argument for the CA case. This became the Feit-Hall-Thompson CN Group Theorem. The entire proof took only sixteen pages and gave Feit and Thompson the courage to attack the general Odd Order Problem, believing it might require about 25 pages to complete. They were off by a factor of 10.

First, Feit and Thompson tried the special case where all Sylow subgroups of *G* are abelian. They were able to reduce the problem to one specific thorny case. Setting this aside, they tackled the entire problem. Fortuitously, Adrian Albert had secured funding for a special year in finite group theory to be held at the University of Chicago in 1960-61. This gave Feit and Thompson the perfect opportunity for intensive collaboration. Charlie Curtis remembers seeing them covering blackboard after blackboard with calculations day and night.

Once again, the structure of the proof is to take a minimal counterexample *G*, which is then a simple group of whose proper subgroups are all solvable. It is still possible to severely restrict the structure of every maximal subgroup *M* of *G*, but it is not quite possible to show that *M* must be a Frobenius group. So-called groups of Frobenius type and 3-step groups must be allowed. This in turn makes the character theoretic analysis much more difficult.

Thompson wrote: "I think there are only a few who understood the precision and subtlety with which Walter handled a variety of character-theoretic situations. Suzuki and, of course, Brauer appreciated Walter's strength. But only Walter and I knew just how intertwined our thinking was over a period of more than a year."

George Glauberman, who was a student at the time, said that he and his contemporaries in group theory thought of Walter as a magician. In a similar vein, when told a decade later that Walter said some problem about linear groups was easy, David Wales replied: "Everything is easy for Walter Feit."

Finally, Feit and Thompson succeeded in reducing the entire problem to the same thorny case that they had encountered when all Sylow subgroups were assumed abelian. In particular, there were a pair *p*, *q* of odd primes satisfying a certain number-theoretic condition. For a while they asked number theorists if they could prove that there were no such primes. This problem remains open, but Thompson was finally able to find an intricate generators-and-relations contradiction. (A somewhat easier argument for this final step was found later by Thomas Peterfalvi.[11]) This completed the proof of:

The Odd Order Theorem. *Every finite group G of odd order is a solvable group.*

The Odd Order Theorem was published in the *Pacific Journal of Mathematics* in 1963.1 It consumed a complete volume of 255 pages. As noted above, Walter and John shared the 1965 Frank Nelson Cole Prize in Algebra in recognition of this paper.

Before completion of the Odd Order Paper, Feit and Thompson published a short paper[12] dedicated to Richard Brauer on the occasion of his sixtieth birthday. Brauer and his student K. A. Fowler had used the fact that two involutions in a finite group *G* generate a dihedral subgroup of *G* to prove that if *G* is a finite simple group of even order with an involution centralizer of size *c*, then $|G| < (c^2)!$ Brauer had then used this as motivation to suggest a strategy for classifying finite simple groups of even order. (In researching a paper on the Theory of Finite Groups in the twentieth century, Feit discovered an 1899 paper of William Burnside that anticipated many of Brauer's and Fowler's methods, much to Brauer's surprise.) Thinking of dihedral groups as polyhedral groups (2, 2, *n*), Feit and Thompson suggested that "other polyhedral groups can surely play a role in group theory which is not totally eclipsed by the groups (2, 2, *n*)." To illustrate this, they used the group (3, 3, 3) to prove:

Theorem. *Let G be a finite group which contains a self-centralizing subgroup of order 3. Then one of the following statements is true:*

(1) *G contains a nilpotent normal subgroup such that $G/N \cong A_3$ or S_3;*
(2) *G contains a normal 2-subgroup N such that $G/N \cong A_5$; or*
(3) *$G \cong PSL(2, 7)$.*

Later in the 1960s, Graham Higman took up this theme. He and several of his students produced some "odd characterizations" of finite simple groups, usually beginning with some hypotheses on the 3-local structure of the group.

In 1959, Jacques Tits introduced the notion of a type of point-line geometry called a generalized *n*-gon for $n \geq 2$, which is characterized by the property that the incidence graph has diameter *n* and girth 2*n*. These are the rank 2 constituents of geometries Tits would call "buildings." They are "thick" geometries whose "thin" skeletons are ordinary *n*-gons. Although finite *n*-gons exist for all $n \geq 2$ (with $n = 2$ a degenerate case), Feit and Graham Higman were able to prove in 1964 that finite generalized *n*-gons only exist for n = 3, 4, 6 or 8 (with $n = 12$ allowed in their more general formulation). Tits refers to this result as "the remarkable theorem of W. Feit and G. Higman" in his treatise on BN-pairs.[13] These pairs are group-theoretic configurations conforming to axioms Tits abstracted from groups of Lie type. They are used to construct buildings, with the rank of the latter identifying with the rank of the BN pair, both identifying with the rank of a Coxeter group *W* (the pair's "Weyl group"). The Feit-Higman theorem, while purely geometric in statement and proof, has as a consequence that finite BN-pairs of rank 2 have Weyl groups, which are dihedral of order 2*n*, with $n = 2, 3, 4, 6$, or 8. The cited Tits volume classifies finite BN-pairs of rank at least 3. Tits gives no such classification for rank 2 but does offer suggestions for strengthening the BN-pair and building axioms to enable such, in each case introducing more group theory or its equivalent. Some of this found its way into the classification of finite split BN-pairs of rank 2 by Fong and Seitz,[14, 15] which also relies heavily on the Feit-Higman theorem.

The ingredients of the Feit-Higman proof were difficult calculations with finite incidence geometries and related matrices, but, beyond that, hard to place. Later, L. Solomon and R. Kilmoyer gave a more conceptual proof[16] in which the calculations are made from the point of view of representations of an algebra with two specified generators. Their argument was soon recast by Donald Higman using his newly formulated theory of coherent configurations, geometrically defined algebras imitating, without groups, endomorphism algebras arising from finite-group permutation modules.[17] At the least, this gave the proof something of a familiar framework.

It was possibly at this point that the theory of generalized n-gons, already known to include finite projective planes as the $n = 3$ case, became part and parcel of the standard landscape of finite geometry. Studies of possible further generalizations of rank 2 buildings were made by William Kantor after providing an outline and overview of the generalized n-gon point of view.[18] Once again, the Feit-Higman result was given high marks as "the most remarkable and important" result in Kantor's outline of generalized n-gon theory because of the restrictions it placed on n.[18] The generality of this result has not been matched, though it has not been sufficient in and of itself for a full rank 2 building theory. The latter was perhaps achieved in 2002, with some concessions (additional hypothesis), in the Tits and Weiss volume on Moufang polygons.[19] The latter theory did, however, allow for infinite point and line sets and was sufficiently effective to be the rank 2 basis of a streamlined version of Tits' earlier work on buildings. A rank 1 BN-pair, without additional hypothesis, is just a doubly transitive group, as Feit remarked in his review of Fong and Seitz.[14,15] Of course, Feit's work on Zassenhaus groups, discussed above, was his principal contribution to the theory of doubly transitive groups. He also wrote several other papers related to finite geometry on topics such as projective planes, error-correcting codes, and block designs.

In 1964, Feit accepted a faculty position at Yale University, where he remained for the rest of his life. He had supervised two Ph.D. students at Cornell University and would have 18 Ph.D. students, including the authors of this article, during his almost 40-year career at Yale.

In 1878, Camille Jordan had proved the existence of a function $J : \mathbb{N} \to \mathbb{N}$ such that every finite subgroup of $GL(n, \mathbb{C})$ contains a normal abelian subgroup A with $|G/A| \leq J(n)$. It should be noted that the normal subgroup A is necessary, since the group D of diagonal matrices in $GL(n, \mathbb{C})$ contains finite (abelian) subgroups of arbitrarily large order. The analogous theorem fails if $\mathbb{C}$ is replaced by $\overline{F}_p$ for p a prime. For example, for all $m \geq 2$, $SL(2, 2^m)$ is a finite simple subgroup of $GL(2, \overline{F}_2)$. In "An Analogue of Jordan's Theorem in Characteristic *p*" (1966),[20] however, Brauer and Feit proved the following conjecture of Otto Kegel:

Theorem. *Let p be a prime. There is an integer-valued function $f_p(m, n)$ such that if K is a field of characteristic p and if G is a finite subgroup of GL(n, K) whose Sylow p-subgroups have order at most p^m, then G has a normal abelian subgroup A with $|G/A| < f_p(m, n)$.*

In 2011, Michael Larsen and Richard Pink published a remarkable improvement[21] of this theorem:

Theorem. *There is a function $J' : \mathbb{N} \to \mathbb{N}$ such that if k is a field of characteristic $p \geq 0$ and G is a finite subgroup of $GL(n, k)$, then G has normal subgroups $G_3 \subseteq G_2 \subseteq G_1$ such that:*

(1) $|G/G_1| \leq J'(n)$;
(2) *Either $G_1 = G_2$ or $p > 0$ and G_1/G_2 is a direct product of finite simple groups of Lie type in characteristic p;*
(3) *G_2/G_3 is abelian of order not divisible by p; and*
(4) *Either $G_3 = 1$ or $p > 0$ and G_3 is a p-group.*

Notice that the Brauer-Feit theorem follows easily from this result together with the fact that if *H* is a finite simple group of Lie type in characteristic *p*, then $|H| < (|H|_p)^3$, where $|H|_p$ is the order of a Sylow *p*-subgroup of *H*. The proof of the Larsen-Pink Theorem relies on deep results in algebraic geometry and algebraic group theory but only elementary results from finite group theory.

Walter remained interested in the classical Jordan result, and in "Finite Linear Groups and Theorems of Minkowski and Schur,"[22] he proved an improvement of related results by Schur and Minkowski concerning finite subgroups of *GL*(*n,* Q). Unpublished work by Boris Weisfeiler, using the Classification of Finite Simple Groups, gave greatly improved versions of Jordan's function $f(n)$. After Weisfeiler's tragic disappearance, Feit encouraged Michael J. Collins to complete Weisfeiler work, which he did, proving in particular in a 2007 paper[23] that for $n \geq 71$, we may take $f(n) = (n + 1)!$ (Note: S_{n+1} is always contained in *GL*(*n*, C). So, this result is the best possible.)

Perhaps stimulated by his work with Brauer on Jordan's Theorem in characteristic *p*, which used Green's theory of vertices and sources, Walter gave a several semester course at Yale in 1967-68 on modular representation theory. His 1969 mimeographed Yale lecture notes for this course evolved into a textbook, *The Representation Theory of Finite*

Groups.[24] Jon Alperin's review notes that this book (together with the earlier Yale notes) was the first comprehensive treatment of the general theory of modular representations of finite groups. Some noteworthy topics that the book treated with great attention included the Green correspondence and its application by Thompson[25] and Dade[26] to create the theory of blocks with cyclic defect group. This work extended a 1941 theory of Brauer treating blocks with defect 1, i.e., defect group of prime order p. (The defect group of a p-block of G is a p-subgroup of G, determined up to G-conjugacy.) Feit's book is widely cited and was translated into Russian in 1990.[27]

Feit made contributions to modular representation theory before, during, and after his book was written. We have already mentioned the Brauer-Feit bound for the number of ordinary irreducible characters in a block.[2] In the modular case, there is no such bound known but a number of conjectures. There are also well-known finiteness conjectures of Donovan and Puig that assert or imply the finiteness of the number of possible Morita equivalence classes of block algebras for finite groups, given their defect groups. Feit has a conjecture, the first version of which he presented at the Santa Cruz Conference on Finite Groups in California in 1979.[28] It was reformulated by Linckelmann in 2018 and called the Feit conjecture.[29] This version states that, for a given field k of characteristic p and a finite p-group P, there are, up to isomorphism, only a finite number of isomorphism classes of finite-dimensional indecomposable kP -modules M with the following property: There is a finite group G and an irreducible kG-module that has P as vertex and M as source. (These notions are central to the Green correspondence mentioned above.) Linckelmann notes that the work of Dade provides a positive answer to this conjecture for M of fixed dimension. Dade's paper[30] even hints that there might be a constructive approach to determining all such source modules M for irreducible kG-modules, which is likely what Feit had in mind.

In 1970, John G. Thompson was awarded the Fields Medal for his extraordinary accomplishments in group theory, most notably the Odd Order Paper and the N-Group Papers. At that ICM, Walter gave a report entitled "The Current Situation in the Theory of Finite Simple Groups."

Feit had a long interest in finite linear groups G (in characteristic 0) of small degree d, either in the absolute sense or in the sense that d is small relative to some prime divisor p of $|G|$, beginning with a joint 1961 paper with Thompson on the case $d < (p - 1)/2$. A sequence of subsequent papers pursued this further for $d < p - 1$. Finally, in 1974, Feit published a paper that both treated a specific but highly interesting $d = p + 1$ case and brought in integral representations.[31] He wrote: "Let G be a finite group. Let K be an algebraic number field contained in the field of complex numbers which is closed under complex conjugation. Let V be a faithful K [G]-module....The object of this paper is to prove some results which assert that under various conditions one can choose a G-invariant lattice in V which has special properties. These properties can then be exploited to give information about the structure of G." In particular, using extensive information about unimodular rational lattices in dimension at most 24, Feit could prove:

Theorem. *Let G be a finite group of order divisible by* 23 *having a faithful rational representation of degree* 24. T*hen either G has a subgroup of index* 23, 24 or, 25, or $G' \leq Co_0$.

Here, Co_0 is the automorphism group of the 24-dimensional Leech lattice. This theorem and a closely related one in the same paper enabled Feit's student, Dan Fendel,[32] to prove that the simple subgroup, Co_3, of the Conway group, Co_0, is the unique simple group having an involution centralizer isomorphic to 2.*Sp*(6, 2), a perfect central extension of the symplectic group *Sp*(6, 2), one of the many essential characterization theorems pursuing Brauer's strategy for the classification of all finite simple groups.

Feit's conjecture in his contribution to the 1981 Santa Cruz Conference on Finite Groups mentioned above was motivated by the approaching completion of the classification of finite simple groups and the opportunities to better understand the modular representation theory of these groups. He began the program of understanding explicitly the Brauer trees of all blocks of finite sporadic simple groups with cyclic defect group,[33] a program continued and partly completed later by Hiss and Lux.[34] Feit also gave general descriptions of graphs with more than 248 vertices that could be the Brauer tree for a block with cyclic defect group. In further work on sporadic groups, Feit completed a program begun by his student Mark Benard to determine all Schur indices for all characters of sporadic groups.[35,36,37] More generally, he championed the conjecture that all Schur indices for characters of finite simple groups are equal to 1 or 2. Much progress on this conjecture was made by Rod Gow, but the conjecture is not yet settled in all cases. Feit also made progress, jointly with his student Leonard Chastkovsky, in determining some block invariants for low-rank groups of Lie type.[38, 39] In particular, they found the degrees of projective indecomposable modules, and explicitly determined one of the Cartan invariants.

As the proof of the classification theorem neared (or so it seemed) its completion in the early 1980s, Thompson turned his attention to the classical problem of determining which finite groups could be realized as Galois groups over the rationals, the so-called

Inverse Galois Problem. Walter Feit was one of the editors of the volume *Proceedings of the Rutgers Group Theory Year, 1983-84*. Notably, he wrote an introduction to the section "Rigidity and Galois Theory"[40] in that volume and contributed, individually and with collaborators, three other articles to that section. All this came on the heels of a dramatic contribution by Thompson. We quote from Walter's introduction:

> *The relevant ideas from algebraic geometry had been known for some time.[41,42,43] However, the full force of these ideas was first exploited by Thompson,[44] who amongst other things isolated the fundamental concept of rigidity and saw it could be used in many explicit cases. As a first application he proved the striking result that the Fischer-Griess monster M is a Galois group over* Q.

Rather than give Thompson's definition of rigidity, we quote from a *MathSciNet* review (by David Surowski) of a paper by Feit and Fong titled "Rational Rigidity of $G_2(p)$ for any prime $p > 5$" and published in the same *Proceedings*[45] that illustrates (a main case of) the rigidity concept and its capability (in the so-called rational rigidity case) of proving existence of realizations of a given group as a Galois group of an extension of Q. Surowski writes:

Let $G = G_2(p)$, where $p \geq 5$ is a prime. Let $\{a, b\}$ be a set of fundamental roots for G with a short and b long. Let C_1 be the class of involutions of G, let C_2 be the class of elements of G containing $x_a(1)$, and let C_3 be the class of elements of G containing $x_a(1)x_b(1)$. Let

$$A = A_G(C_1, C_2, C_3) = \{(x_1, x_2, x_3) : x_1x_2x_3 = 1, x_i \in C_i\}.$$

The result proved in the papers under review is that A is rationally rigid. In other words, the following three conditions hold:

(1) If $x_i \in C_i$ and if χ is a complex character of G, then $\chi(x_i)$ *is an integer.*

(2) $|A| = |G|$.

(3) If $(x_1, x_2, x_3) \in A$, then $G = \langle x_1, x_2, x_3 \rangle$.

Conditions (2) and (3) define the rigidity of the set A of triples, and these conditions are already enough (at least for simple groups) to insure G is a Galois group over an abelian extension of Q. Condition (1), in the presence of the other two conditions, is a formulation of the rationality property that allows the extension to be taken over Q itself.

The rational rigidity of $G_2(5)$ had been proved by Thompson. Using the classification of finite simple groups, Feit and Fong were able to verify Condition (1) for all $p > 5$. As might be expected, the proofs of Feit-Fong are more character theoretical than Thompson's, not only in checking item (2), but also in eliminating possible overgroups of x_1, x_2, x_3 in checking (3) by using, in part, class algebra structure constants. In their 1999 book, Malle and Matzat give both Thompson and Feit-Fong independent credit for the case $p \geq 5$ for $G_2(p)$ but adapt the Feit-Fong proof for their general approach to treating groups $G(p)$ of twisted or untwisted exceptional Lie type. These groups $G(p)$ are treated as an interesting "family" much in the same spirit as the "family" of sporadic groups. Naturally enough, there were (and are) fewer open cases in the sporadic simple groups family (only the Mathieu group M_{23} remains). Information on known and unknown cases for $G(p)$ is available (in all types) in Malle and Matzat's book.[46]

When one takes a step away from simple groups, the next groups to handle are their automorphism groups and central extensions. For simple groups G (or, more generally, groups with trivial center), rigidity methods that exhibit G as a Galois group over Q often extend to show Aut(G) is also a Galois group over Q. Malle and Matzat provide a discussion and a list in their 1999 monograph.[46] Passing to central extensions is another story; the rigidity theory of G does not apply so directly to its covering groups as to its automorphism groups, and much less is known. For a while, only central extensions using central groups of order 2 were studied. Feit initiated the study of 2-fold central extensions of the simple alternating groups, showing that $2A_n$ had such a realization at least for $n \equiv 3 \pmod 4$.[47] He used Serre's 1984 paper,[48] which formulated a cohomological obstruction theory especially suitable for the 2-fold case, where it could be formulated in terms of a quadratic trace form. Returning to rigidity methods, Feit introduced some new approaches to realizing central extensions, especially effective in the 3-fold cases, in his 1987 article.[49] Informally, he cleverly adds an involution to the covering group to act on the order 3 center, thus killing it in some sense, without actually removing it. Then he applies rigidity methods to the larger group, recovering the intended central covering as a normal subgroup of index 2 (in the spirit of Theorem C in his 1985 contribution to the Rutgers Group Theory Year Proceedings,[40]) throwing away the added involution. Some of his results are summarized by Malle and Matzat[46] in the theorem below, attributed to Feit's results in his 1989 article.[49] In their preamble to the theorem, they give a more formal summary of the proof idea, stating that it is "to obtain central Frattini extensions as subgroups of centerless Frattini extensions." The terminology G-realization over Q below is technical, referring to a (strong) way a given group can be realized as a Galois group over Q by first being realized through an extension of function fields over Q, then applying Hilbert's Irreducibility Criterion.

Theorem. *(a) The central group extensions of types $3A_6$ and $3A_7$ possess G-realizations over* Q.

(b) The covering groups $3 \cdot S$ with $S \in \{M_{22}, McL, Suz, ON, Fi_{22}, Fi'_{24}\}$ possess G-realizations over Q.

Malle and Matzat remark that the groups in part (b) contain, with the exception of J_3, all sporadic groups whose Schur multipliers have orders divisible by 3. We can add that the groups A_6, A_7 in part (a) are the only simple alternating groups with Schur multipliers having this divisibility property. In general, the sporadic groups and the alternating groups have cyclic Schur multipliers with orders divisible only by the primes 2 and 3. And these multipliers are sometimes trivial in the sporadic group cases, so that the above theorem, especially with the earlier work on order 2 coverings, is closer to a picture of central extension realizations for these two families (sporadic and alternating) than might be guessed at first glance. In the alternating group family, the existence of central extension realizations is now known in all cases. This is mostly thanks to later work of Mestre that treats 2-fold central extensions of all of the simple alternating groups.[50] There are also two 3-fold central extensions, handled by part (a) of the theorem above, and two 6-fold extensions $6A_6$ and $6A_7$. Both are treated by Feit in his 1989 article,[49] an announcement by Mestre on the first page of the latter paper, facilitated through a personal communication with Feit.

Walter Feit continued to publish articles on inverse Galois theory for the rest of his mathematical career, but the papers discussed briefly above represent some of his most significant contributions, selected from eight publications in the period 1985-89 of rapid evolution of the subject. In 1990, a conference was held at Oxford University and a two-issue volume of the *Journal of Algebra* was published to honor his sixtieth birthday. In addition to many articles on or related to finite group representations and characters were many articles on or related to Galois theory. These included Mestre's article cited above, as well as articles individually authored by G. Malle, D. Saltman, J. Sonn, N. Villla, and J. Walter. There were also joint articles by B. Fein and M. Schacher, R. Foote, and D. Wales, and R. Guralnick and J. Thompson. Many of the Galois theory authors had been invited to submit a paper at the suggestion of J.-P. Serre, who had often corresponded with Walter regarding Galois theory.

In addition to Feit's students, many notable algebraists studied with Walter in a postdoctoral or junior faculty capacity, including Don Passman, Bernd Fischer, Larry Dornhoff, David Goldschmidt, Richard Lyons, David Sibley, and Dave Benson.

Walter was a charming conversationalist, a lover of good food, and a very knowledgeable history buff. In the words of Dan Mostow, "He knew, in detail, the history of every country, ancient or modern, as far as I could tell." In addition to his lifelong friendship with John G. Thompson, Walter had many dear friends in the community of algebraists of his generation, including J.-P. Serre, Jacques Tits, and Bob Steinberg. He also earned the veneration of the group and representation theorists of succeeding generations. The great success of a conference held at Yale University in October 2003 to honor his career on the occasion of his retirement from the Yale faculty was a testament to this admiration and affection. Sadly, cancer would claim Walter's life just a few months later on July 29, 2004.

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