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KURT GÖDEL

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A Biographical Memoir by

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Biographical Memoir

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Kurt Gödel

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BY STEPHEN C. KLEENE[1]

TWO PAPERS (1930a, 1931a), both written before the author reached the age of twenty-five, established Kurt Gödel as second to none among logicians of the modern era, beginning with Frege (1879).[2] A third fundamental contribution followed a little later (1938a, 1938b, 1939a, 1939b).

ORIGINS AND EDUCATION, 1906–1930

Gödel was born at Brünn in the Austro-Hungarian province of Moravia. After World War I, Brünn became Brno in Czechoslovakia. Gödel's father Rudolf was managing director and partial owner of one of the leading textile firms in Brünn; his family had come from Vienna. His mother Marianne had a broad literary education. Her father, Gustav Handschuh, had come from the Rhineland. Gödel's family cultivated its German national heritage.

After completing secondary school at Brno, in 1924 Gödel went to Vienna to study physics at the university. The elegant lectures of P. Furtwängler (a pupil of Hilbert and cousin of the famous conductor) fed his interest in mathe-

[1] For some details of Gödel's life, I have drawn upon Kreisel (1980) and Wang (1978, 1981); the authors kindly supplied me with copies.

[2] A date shown in parentheses refers to a work listed in the References (or for Gödel, in the Bibliography), under the name of the adjacent author.

matics, which became his major area of study in 1926. His principal teacher was the analyst Hans Hahn, who was actively interested in the foundations of mathematics. Hahn was a member of the Vienna Circle (*Wiener Kreis*), the band of positivist philosophers led by M. Schlick, who was assassinated during a lecture in 1936. Gödel attended many of their meetings, without subscribing to all of their doctrines (see Wang 1981, 653). His doctoral dissertation was completed in the autumn of 1929, and he received the Ph.D. on February 6, 1930. A somewhat revised version was presented at Karl Menger's colloquium on May 14, 1930, and was published (1930a) using "several valuable suggestions" of Professor Hahn "regarding the formulation for the publication" (Wang 1981, 654; 1930b is an abstract).

GÖDEL'S COMPLETENESS THEOREM (1930a)

Since Frege, the traditional subject-predicate analysis of the structure of sentences has been replaced by the more flexible use of *propositional functions*, or, more briefly, *predicates*.[3] Using a given collection—or *domain* D—of objects (we call them *individuals* if they are the primary objects not being analyzed) as the range of the independent variables, a one-place predicate P or $P(a)$ over D (also called a *property* of members of D) is a function that, for each member of D as value of the variable a, takes as its value a proposition $P(a)$. A two-place predicate Q or $Q(a,b)$ (also called a *binary relation* between members of D), for each pair of values of a and b from D, takes as value a proposition $Q(a,b)$; and so on. In the most commonly cultivated version of logic (the *classical* logic), the

[3] I am endeavoring to give enough background material to enable a scientist who is not a professional mathematician and not already acquainted with mathematical logic to understand Gödel's best-known contributions. The memoir (1980) by Kreisel, about three times the length of the present one, includes many interesting details addressed to mathematicians, if not just to mathematical logicians. The memoir (1978) by Quine gives an excellent overview in just under four pages.

propositions taken as values of the predicates are each either *true* or *false*.

The *restricted* or *first-order predicate calculus* ("elementary logic") deals with expressions, called *formulas*, constructed, in accordance with stated syntactical rules, from: variables a, b, c, ..., x, y, z for individuals; symbols P, Q, R, S, ... for predicates; the propositional connectives $\neg$ ("not"), & ("and"), $\vee$ ("or") and $\rightarrow$ ("implies"); and the quantifiers $\forall x$ ("for all x") and $\exists x$ ("[there] exists [an] x [such that]"). For example, taking P, Q, R to be symbols for predicates of one, two, and three variables, respectively, the expressions P(b), Q(a,c), R(b,a,a), $\forall x P(x)$, $\forall x \exists y Q(x,y)$, $\forall x((\neg P(x)) \rightarrow Q(a,x))$, and $\forall x((\exists y Q(y,x)) \rightarrow \neg R(x,a,x))$ are formulas.

In the classical logic, after making any choice of a non-empty domain *D* as the range of the variables, each formula can be evaluated as either true or false for each *assignment in D* of a predicate over *D* as the value or interpretation of each of its predicate symbols, and of a member of *D* as the value of each of its "free" variables. Its *free* variables are the ones with "free" occurrences, where they are not operated upon by quantifiers. In the seven examples of formulas given above, the eight occurrences of a, b, and c are free; the fifteen occurrences of x and y are not free, that is, they are *bound*. The evaluation process is straightforward, taking $\vee$ to be the inclusive "or" (A $\vee$ B is true when one or both of A and B are true, and false otherwise), and handling A $\rightarrow$ B like ($\neg$A) $\vee$ B. For example, taking *D* to be the non-negative integers or *natural numbers* {0, 1, 2, ...}, and assigning to P(a), Q(a,b) and R(a,b,c) the predicates "*a* is even", "*a* is less than *b*" and "$ab = c$", and to a, b, and c the numbers 0, 1, and 1, as values, our seven examples of formulas are respectively false, true, true, false, true, true, and true.

Logic is concerned with exploring what formulas express logical truths, that is, are "true in general". Leibnitz spoke of truth in all possible worlds. We call a formula *valid in D* (a given non-empty domain) if it is true for every assignment in *D;* and simply *valid* if it is valid in every non-empty domain *D*.

To make reasoning with the predicate calculus practical, paralleling the way we actually think, we cannot stop to think through the evaluation process in all non-empty domains for all assignments each time we want to assure ourselves that a formula is logically true (valid). Instead we use the "axiomatic-deductive method", whereby certain formulas become "provable".

First, certain formulas are recognized as being *logical axioms.* For example, all formulas of either of the following two forms—the forms being called *axiom schemata*—are axioms:

$$A \rightarrow (B \rightarrow A). \qquad \forall x A(x) \rightarrow A(a).$$

Here A, B, and A(x) can be any formulas, and x and a any variables; A(a) is the result of substituting the variable a for the free occurrences of the variable x in the formula A(x). Furthermore, it is required that the resulting occurrences of a in A(a) be free; thus $\forall x \exists b Q(b,x) \rightarrow \exists b Q(b,a)$ is an axiom by the schema $\forall x A(x) \rightarrow A(a)$, but $\forall x \exists a Q(a,x) \rightarrow \exists a Q(a,a)$ is not. The axiom schemata (and particular axioms, if we have some not given by schemata) are chosen so that each axiom is valid.

Second, circumstances are recognized, called *rules of inference,* in which, from the one or two formulas shown above the line called *premises,* the formula shown below the line called the *conclusion* can be *inferred;* for example:

$$\frac{A, \qquad A \rightarrow B}{B.} \qquad \frac{C \rightarrow A(x)}{C \rightarrow \exists x A(x).}$$

Here A, B, and A(x) can be any formulas, x any variable, and C any formula not containing a free occurrence of x. The rules of inference are chosen so that, whenever for a given non-empty domain *D* and assignment in *D* the premises are true, then for the same *D* and assignment the conclusion is true. Hence, if the premises are valid, the conclusion is valid.

A *proof* is a finite list of formulas, each one in turn being either an axiom or the conclusion of an inference from one or two formulas earlier in the list as the premise(s). A *proof* is a *proof of* its last formula, which is said to be *provable.*

In one of the standard treatments of the classical first-order predicate calculus (Kleene 1952, 82), twelve axiom schemata and three rules of inference are used.[4]

By what we have just said about how the axiom schemata (or particular axioms) are chosen (so each axiom is valid), and likewise the rules of inference (so truth is carried forward by each inference), *every provable formula is valid.* Thus the axiomatic-deductive treatment of the predicate calculus is *correct.*

But is it *complete*? That is: *Is every valid formula provable*?

The axiomatic-deductive treatment of the first-order predicate calculus, separated out from more complicated logical systems, was perhaps first formulated explicitly in Hilbert and Ackermann's book (1928). The completeness problem was first stated there (p. 68): "Whether the system of axioms [and rules of inference] is complete, so that actually all the logical formulas which are correct for each domain of individuals can be derived from them, is still an unsolved question."

It is this question that Gödel answered in (1930a). He established: *For each formula* A *of the first-order predicate calculus, either* A *is provable in it, or* A *is not valid in the domain* {0, 1, 2, ...} *of the natural numbers* (and therefore is not valid).

So, *if* A *is valid, then* Gödel's second alternative is excluded,

[4] I am giving the version of the predicate calculus with predicate symbols instead of predicate variables, after von Neumann (1927). This I consider easier to explain.

and A *is provable.* This answers Hilbert and Ackermann's question affirmatively.

Let us say that a formula A is (or several formulas are simultaneously) *satisfiable* in a given domain *D* if A is satisfied (all the formulas are satisfied), that is, made true, by some assignment in *D*. Then A-is-satisfiable-in-*D* is equivalent to ¬A-is-not-valid-in-*D*.

Now *if* A *is satisfiable in some domain D, then* ¬A is not valid in that domain *D,* so ¬A is not valid, so ¬A is not provable (by the correctness of the predicate calculus), so by Gödel's result applied to ¬A, ¬A is not valid in {0, 1, 2, ...}, so A *is satisfiable in* {0, 1, 2, ...}. This is a theorem of Löwenheim (1915).

Restating the completeness theorem for ¬A: *Either* ¬A is not valid (that is, A *is satisfiable*) *in* {0, 1, 2, ...}, *or* ¬A *is provable* (equivalently, a contradiction can be deduced from A).

Gödel also treated the case for an infinite collection of formulas $\{A_0, A_1, A_2, ...\}$ in place of one formula. The result is just what comes from substituting $\{A_0, A_1, A_2, ...\}$ for A in the immediately preceding statement, and noting that, if a contradiction can be deduced from the formulas A_0, A_1, A_2, ..., only a finite number of them can participate in a given deduction of the contradiction. Thus: *Either the formulas* A_0, A_1, A_2, ... *are simultaneously satisfiable in* {0, 1, 2, ...}, *or, for some finite subset* $\{A_{i_1}, ..., A_{i_n}\}$ *of them,* $\neg(A_{i_1} \ \& \ ... \ \& \ A_{i_n})$ *is provable* (and hence valid, so A_{i_1}, ..., A_{i_n} are not simultaneously satisfiable in any domain).

Now, *if the formulas of each finite subset of* $\{A_0, A_1, A_2, ...\}$ *are simultaneously satisfiable in a respective domain, then* the second alternative just above is excluded, so *all the formulas are simultaneously satisfiable* (this result is called "compactness"), *indeed in the domain* {0, 1, 2, ...} (the Löwenheim-Skolem theorem). Skolem in (1920), in addition to closing up a gap in Löwenheim's (1915) reasoning, added the case of infinitely many formulas.

These satisfiability results, which are coupled with the completeness theorem in Gödel's treatment, have surprising consequences in certain cases when we aim to use a collection of formulas as axioms to characterize a mathematical system of objects. In doing so, the formulas are not to be logical axioms, but rather mathematical axioms intended to be true, for a given domain D and assignment of predicates over D to the predicate symbols, exactly when D and the predicates have the structure we want the system to have. To make the evaluation process apply as intended, I shall suppose the axioms to be *closed*, that is, to have no free variables.

In using the symbolism of the predicate calculus to formulate mathematical axioms, we usually want to employ a predicate symbol E(a,b) intended to express $a = b$, and usually written a=b. Then, for our evaluation process we are only interested in assignments that give E(a,b) the value $a = b$, that is, that make E(a,b) true exactly when a and b have the same member of D as value. Adding some appropriate axioms to the predicate calculus for this case (Kleene 1952, top 399), we get the *first-order predicate calculus with equality*. Gödel offered supplementary reasoning that adapted his treatment for the predicate calculus to the predicate calculus with equality, with "the domain {0, 1, 2, ...}" being replaced in his conclusions by "{0, 1, 2, ...} or a non-empty finite domain".

Cantor in (1874) established that the set of all the subsets of the natural numbers {0, 1, 2, ...} (or the set of the sets of natural numbers) is more numerous, or has a *greater cardinal number*, than the set of the natural numbers. To explain this, I review some notions of Cantor's theory of sets. He wrote (1895, 481): "By a 'set' we understand any collection M of definite well-distinguished objects m of our perception or our thought (which are called the 'elements' [or 'members'] of M)

into a whole." A set N is a *subset* of a set M if each member of N is a member of M. For example, the set {0, 1, 2} with the three members shown has the following 8 (= 2^3) subsets: {0, 1, 2}, {1, 2}, {0, 2}, {0, 1}, {0}, {1}, {2}, { }. Cantor showed that there is no way of pairing all the sets of natural numbers (that is, all the subsets of the natural numbers) with the natural numbers, so that one subset is paired with 0, another with 1, still another with 2, and so on, with every natural number used exactly once. Sets have the *same cardinal number* if they can be thus paired with each other, or put into a "one-to-one correspondence". Denoting the cardinal number of the natural numbers by $\aleph_0$, and adopting $2^{\aleph_0}$ as a notation for the cardinal of the sets of natural numbers, thus $2^{\aleph_0} \neq \aleph_0$. But the natural numbers can be paired with a subset of the sets of natural numbers (indeed with the *unit sets* {0}, {1}, {2}, ... having one member each), so we write $2^{\aleph_0} > \aleph_0$.

Sets that are either finite or have the cardinal $\aleph_0$ are called *countable;* other sets, *uncountable.* The real numbers (corresponding to the points on a line) have the same cardinal $2^{\aleph_0}$ as the sets of natural numbers.

I have been tacitly assuming for the first-order predicate calculus (without or with equality) that only a countable collection of variables and of predicate symbols is allowed. This entails that only a countable collection of formulas exists.

Now suppose that we want to write a list of formulas A_0, ..., A_n or A_0, A_1, A_2, ... in the first-order predicate calculus with equality to serve as axioms characterizing the sets in some version of Cantor's theory of sets. Presuming that the axioms are satisfied simultaneously in some domain D (the "sets" in that version of Cantor's set theory) by some assignment (the one understood in his theory), it follows by the Löwenheim-Skolem theorem that they are also satisfiable in the countable domain {0, 1, 2, ...}! (It is evident that they are not satisfiable in a finite domain.) That is, one can so interpret the axioms that the range of the variables in them constitutes a countable collection, contradicting the theorem of Cantor by which the subsets of {0, 1, 2, ...} (which are among the sets for his theory) constitute an uncountable collection. This is Skolem's paradox (1923). It is not a direct contradiction; it only shows that we have failed by our axioms to characterize the system of all the sets for Cantor's theory, as we wished to do.

Suppose instead that we want a list of formulas A_0, A_1, A_2, ... in the first-order predicate calculus with equality to serve as axioms characterizing the system of the natural numbers 0, 1, 2, ... Skolem in (1933, 1934) showed that we cannot succeed in this wish. He constructed so-called "non-standard models of arithmetic", mathematical systems satisfying all the axioms A_0, A_1, A_2, ... (or indeed all the formulas that are true in the arithmetic of the natural numbers) but with a different structure (as the mathematicians say, not *isomorphic* to the natural numbers). In fact, as seems to have been noticed first by Henkin in (1947), the existence of non-standard models of arithmetic is an immediate consequence of the compactness part of Gödel's completeness theorem for the predicate calculus with equality.

Before giving Henkin's argument, I observe that we may enlarge the class of formulas for the first-order predicate calculus with equality by allowing *individual symbols* i, j, k, ..., which for any assignment in a domain *D* are given members of *D* as their values, and *function symbols* f, g, h, ..., where, for example, if f is a symbol for a two-place function, its interpretation in any assignment is as a function of two variables ranging over *D* and taking values in *D*. Examples that come to mind for systems of axioms for the natural numbers are 0 as an individual symbol (with the number 0 as its standard interpretation), ′ as a one-place function symbol (to be interpreted by +1), and + and × as two-place function symbols (for addition and multiplication). Such additions to the symbolism are not essential. We could equivalently use predicate symbols Z(a), S(a,b), A(a,b,c), and M(a,b,c), where Z(a) is taken as true exactly when the value a of a is 0; S(a,b) when $a+1 = b$; A(a,b,c) when $a+b = c$; and M(a,b,c) when $ab = c$. Here "equivalently" means that whatever can be expressed using the individual and function symbols can be paraphrased using the predicate symbols (but at a considerable loss of convenience). This is shown in Hilbert and Bernays (1934, 460 ff.) and Kleene (1952, §74).

Now take the proposed list of axioms A_0, A_1, A_2, ..., which are true under the interpretation by the system of the natural numbers. I shall assume they include (or if necessary add to them) the axioms $\forall x(x' \neq 0)$ and $\forall x \forall y(x' = y' \rightarrow x = y)$. Now consider instead the list A_0, $\neg i = 0$, A_1, $\neg i = 0'$, A_2, $\neg i = 0''$, ... where i is a new individual symbol. Each finite subset of these formulas is true under the intended interpretation of the old symbols and interpreting i by a natural number n for which $\neg i = 0^{(n)}$ (with n accents on the 0) is not in the subset. So by compactness, A_0, $\neg i = 0$, A_1, $\neg i = 0'$, A_2, $\neg i = 0''$, ... are simultaneously satisfiable. It is easy to see that the satisfying system is isomorphic to one in which 0, 1, 2, ... are the values of 0, 0′, 0″, ... and the value of i is not a natural number—a non-standard model of arithmetic.

These illustrations will suggest the power of Gödel's completeness theorem (1930a) with its corollaries as a tool in studying the possibilities for axiomatically founding various mathematical theories.

Actually, not only was the Löwenheim-Skolem theorem around earlier than 1930, but it has been noticed in retrospect that the completeness of the first-order predicate calculus can be derived as an easy consequence of Skolem (1923). Nevertheless, the possibility was overlooked by Skolem himself; indeed the completeness problem was first formulated in Hilbert-Ackermann (1928). Skolem worked with logic intuitively rather than using an explicitly described set of axioms and rules of inference. Gödel's treatment of the problem in (1930a) was done without knowledge of Skolem (1922), which Hilbert and Ackermann do not mention, and was incisive, obtained the compactness, and included the supplementary argument to make it apply to the predicate calculus with equality.

VIENNA, WITH VISITS TO PRINCETON (IAS) 1930–1939

Gödel's father died in 1929, and Gödel's mother moved to Vienna. She took a large flat and shared it with her two sons, until she returned to her beautiful villa in Brno in 1937. Rudolf, the elder son, was already a successful radiologist in Vienna. The theater in Vienna appealed to her literary interests, and the sons went with her.

Gödel began in 1930 to work on the consistency problem of Hilbert's formalist school, which I will describe in the next section. His approach to this (as described in Wang (1981, §2)) led him to some results on undecidable propositions (preliminary to 1931a), which he announced at a meeting in September 1930 at Königsberg (1930c). von Neumann was much interested and had some penetrating discussions with Gödel, both at the meeting and by correspondence. In November 1930, Gödel's famous paper (1931a) was completed and sent to the *Monatshefte* (received November 17, 1930). It was accepted by Hahn as Gödel's *Habilitationsschrift* on January 12, 1932. (1930d) and (1930e) are abstracts of it and (1931d) is relevant to it.

From 1931 through 1933 Gödel attended Hahn's seminar on set theory (Hahn died in 1934), and took part in Karl Menger's colloquium, which yielded proceedings that reported a number of Gödel's results. In 1933 he was appointed a *Privatdozent* (an unpaid lecturer) at Vienna. In the academic year 1933–34, he went to Princeton as a visitor at the Institute for Advanced Study, and lectured on his (1931a) results; Rosser and I took the notes (1934).

He again visited the IAS in the fall of 1935. While he was there (according to Kleene (1978) and Wang (1981, Footnote 7)), he told von Neumann of his plan for proving the relative consistency of the axiom of choice and the continuum hypothesis by use of his concept of "constructible sets". He completed his plan three years later (1938a, 1938b, 1939a, 1939b), as will be discussed in the second section below.

On September 20, 1938, he married Adele Porkert. (She survived him by three years, passing away on February 4, 1981.) He returned to the IAS in Princeton in the fall of 1938. In the spring of 1939 he lectured at Notre Dame, and he returned to Vienna in the fall of 1939.

Gödel's mother, almost alone among her friends and neighbors, had been skeptical of the successes of Germany under Hitler. In March 1938, when Austria became a part of Germany and the title of *Privatdozent* was abolished, Gödel was not made a *Dozent neuer Ordnung,* (paid) lecturer of the new order, as were most of the university lecturers who had held the title of *Privatdozent.* He was thought to be Jewish, and once for this reason he was attacked in the street by some rowdies. Concerning his application of 25 September 1939 for a *Dozentur neuer Ordnung,* the *Dozentenbundsführer* wrote on 30 September 1939 (without supporting or rejecting the application) that Gödel was not known ever to have uttered a single word in favor of or against the National Socialist movement, although he himself moved in Jewish-liberal circles, though with mitigating circumstances. (The application was actually accepted on 28 June 1940, when Gödel was no longer available.) Gödel was bitterly frustrated. He was apprehensive that he might be conscripted into the German army, despite his frail health, which he believed rendered him unfit for military service. So at the end of 1939, he returned to Princeton, crossing the U.S.S.R. on the Trans-Siberian Railway.

As a sequel, his mother stayed at her villa in Brno. She was openly critical of the National Socialist regime (thereby losing most of her former friends), so she did not expect reprisals by the Czechs. She returned to Vienna in 1944. But after the war, under the treaty between the Austrian government and Czechoslovakia, she received for her villa only a tenth of its assessed value.

GÖDEL'S INCOMPLETENESS THEOREMS (1931a), ETC.

Cantor's development of set theory, begun in (1874), had led—beginning in 1895—to the discovery of paradoxes in it by himself, Cesari Burali-Forti, Bertrand Russell, and Jules Richard. For a quick illustration, I state the Russell paradox (for the others, and references, see Kleene (1952, §11)). Russell considered the set T of all those sets that are not members of themselves, which seemed to come under Cantor's definition of 'set', quoted above. Is T a member of T? In symbols, does $T \varepsilon T$? Suppose $T \varepsilon T$; then by the definition of T, not $T \varepsilon T$ (in symbols $T \not\varepsilon T$), contradicting the supposition. So by reductio ad absurdum, $T \not\varepsilon T$. Similarly, supposing $T \not\varepsilon T$, $T \varepsilon T$. Thus both $T \not\varepsilon T$ and $T \varepsilon T$!

The appearance of the paradoxes gave special impetus to thinking about the foundations of mathematics, beyond what was already called for by the very extensive reformulations of various branches of mathematics in the nineteenth century. By the mid-1920s, three principal schools of thought had evolved.

The *logicistic* school was represented by Bertrand Russell and Alfred North Whitehead. It proposed to make mathematics a branch of logic, in accordance with Leibnitz's 1666 conception of logic as a science containing the ideas and principles underlying all other sciences. They proposed to deduce the body of mathematics from logic, continuing from work of Frege, Dedekind, and Peano (see Kleene 1952, 43–46). To avoid the newly discovered paradoxes, Russell formulated his theory of types (1908), in which the individuals (or primary objects not being subjected to analysis) are assigned to the lowest type 0, the properties of individuals (or one-place predicates over type 0) to type 1, the properties of type-1 objects to type 2, and so on. A rather definite structure was assumed for the totality of the possible definitions of objects of a given type. The deduction on this basis of a very large portion of the existing mathematics was carried out in the monumental *Principia Mathematica* (PM) of Whitehead and Russell in three volumes (1910, 1912, 1913).

Neither of the other two schools, the *intuitionistic* and the *formalistic,* agreed to start back in logic to deduce the simplest parts of mathematics, such as the elementary theory of the natural numbers 0, 1, 2, ... Indeed, it can be argued that mathematical conceptions on this level are already presupposed in the formulation of logic with the theory of types.

The *intuitionistic* school of thought dates from a paper of Brouwer (1908) criticizing the prevailing or "classical" logic and mathematics. Brouwer argued that classical logic and mathematics go beyond intuition in treating infinite collections as actually existing. As an example, each of the natural numbers 0, 1, 2, ... is a finite object; but there is no last one. Mathematicians can often establish that a property is possessed by every natural number n by reasoning that involves working with only the natural numbers out to a certain point depending on n (maybe just with the numbers $\leq n$). Thus the infinity is only a *potential* infinity (an horizon within which we work). On the other hand, much of the existing classical mathematics deals with infinite collections as completed or *actual* infinities. Some reasoning with the natural numbers uses an actual infinite; for example, the application of the law of the excluded middle to say that either some natural number has a certain property *P*, or that is not the case (so every natural number has the property not-*P*). The use of infinite collections as actual infinities is pervasive in the usual theory of the real numbers, represented say using infinite decimals. Brouwer, in papers beginning in 1918 (exposition in Heyting 1971), proposed to see how far mathematics could be redeveloped using only methods that he considered intuition as justifying: that is, methods using only potential infinities, not actual ones. Brouwer was able to go rather far in this direction, at the cost of altering the subject substantially from the classical form as typified by the classical analysis that physicists are accustomed to applying.

The *formalistic* school was initiated by Hilbert in (1904), and he developed it with a number of collaborators after 1920. Hilbert agreed with the intuitionists that much of classical mathematics goes beyond intuitive evidence. He drew a distinction between *real* statements in mathematics, which have an intuitive meaning, and *ideal* statements, which do not but in classical mathematics are adjoined to the real ones to make mathematical theories simpler and more comprehensive. His real statements are those that correspond to the use of infinity only potentially, while an actual infinite is involved in the ideal statements. But rather than simply abandoning the ideal parts of mathematics, Hilbert had another proposal.

We saw above how the first-order predicate calculus, after logical propositions are expressed as formulas in a precisely regulated symbolic language, was organized by the axiomatic-deductive method. Whitehead and Russell, and Hilbert, proposed to do the same for mathematics generally, that is for very substantial portions of mathematics short of the paradoxes. As we saw, Whitehead and Russell proposed to make all of it logic, but not just first-order logic, within which mathematics is to be defined. Instead, Hilbert proposed to start with mathematical axioms as well as logical axioms. This can be done in the symbolism of the first-order predicate calculus, or using a second-order predicate calculus (with quantification of properties of individuals), or still higher-order predicate calculi. In proofs in a system obtained by adding mathematical axioms to the logical apparatus of the first-order predicate calculus (or, as we may call them, "deductions" by logic from the mathematical axioms), we are exploring formulas that are true for each domain *D* and assignment in *D* that satisfy the axioms. A symbolic language is first established with an exactly specified syntax (thus, a class of *formulas*), and then an exactly defined concept of *proofs* (by starting with *axioms,* logical or mathematical, and applying *rules of inference*). We call the result a *formal system.* (The first-order predicate calculus as described above is a formal system with only logical axioms.)

For Whitehead and Russell, our confidence in the result—the deduction of mathematics within PM—was to rest on our being convinced of the correctness of the logical principles embodied in their version of logic, inclusive of the theory of types, from which all the rest is deduced.

Hilbert proposed to "formalize" one or another mathematical theory, and he hoped to continue with the whole body of mathematics up to some point short of encountering the paradoxes, in formal systems. Typically, the mathematics formalized will in part be ideal and thus not supported by our intuitions. Then he wanted to look at such a system from outside. The formal system, looking just at its structure (apart from the meanings or supposed meanings expressed by the symbols, which guide the practicing mathematician) is a system of finite objects: symbols (from an at most countably infinite collection), finite sequences of symbols (like those constituting formulas), and finite sequences of finite sequences of symbols (like those constituting proofs). So there is the possibility of applying to the study of a formal system intuitive methods of reasoning in the real part of mathematics (using only potential infinities), which Hilbert called *finitary* (German *finit*).

In particular, Hilbert hoped by finitary reasoning to prove the *consistency* of each of his formal systems, that is, that no two proofs in it can end in a pair of contradictory formulas A and ¬A. This would show that mathematics, as it has been developed classically by adjoining the ideal statements to the real ones, is not getting into trouble. Thus Hilbert proposed to give a kind of justification to the cultivation of those parts of classical mathematics that the intuitionists reject. The mathematical discipline in which formal systems (often embodying ideal mathematics) are studied from outside in respect to their structure (leaving out of account the meanings of the symbols) as part of real mathematics, using only finitary methods, Hilbert called *proof theory* or *metamathematics.* Full-length expositions are in Hilbert and Bernays (1934, 1939) and Kleene (1952).

Now we are in a position to understand Gödel's (1931a) results.

Clearly, having embodied some part of mathematics in a formal system, a question of completeness arises just as we saw for the formal system of the first-order predicate calculus.

Specifically, Gödel considered formal systems like that of *Principia Mathematica* and systems constructed by the formalists that aim to formalize at least as much of mathematics as the elementary theory of the natural numbers. (A formal system that didn't do this much would be of rather little interest for the programs of the logicistic and formalistic schools.)

In such a system, propositions of elementary number theory can be expressed by closed formulas, that is, ones containing no free variables. Completeness should then mean that, for each closed formula A, either A itself or its negation, $\neg$A, is provable. That is, for the system to be complete, proofs in the system should provide the answer "yes" (A is provable) or "no" ($\neg$A is provable) to any question about natural numbers "Is the proposition *P* true?" such that *P* can be expressed, under the intended meaning of the symbols, by a closed formula A. For example, with the variables interpreted to range over the natural numbers, if A(x,y) is a formula (with only the free variables x and y) expressing $x < y$, one of the two closed formulas $\forall x \exists y A(x,y)$ and $\neg \forall x \exists y A(x,y)$ should be true—indeed the first is—and this one should be provable if the formal system is complete. The *open* formulas $\exists x A(x,y)$ and $\neg \exists x A(x,y)$, in ordinary usage, are synonymous with their *closures* $\forall y \exists x A(x,y)$ and $\forall y \neg \exists x A(x,y)$, and neither is true.

Gödel's first incompleteness theorem of (1931a)—famous simply as "Gödel's theorem"—says that a formal system *S* like that described, if correct, is incomplete. *There is in S a closed formula* G *such that, if in S only true formulas are provable, then neither* G *nor* $\neg$G *is provable in S (although indeed under the intended interpretation* G *is true).*

To be more specific about the assumption of correctness, let us take into account the form of G, which is $\forall x A(x)$, where for the interpretation the intended range of the variable x is the natural numbers. Here A(x) is a formula with the following property. Let us substitute in A(x) for the free occurrences of the variable x successively the expressions (called *numerals*) $0, 0', 0'', \ldots, \boldsymbol{x}, \ldots$ which express the natural numbers $0, 1, 2, \ldots, x, \ldots$. I am denoting the numeral for x by "$\boldsymbol{x}$", and I write the result of the substitution as "A($\boldsymbol{x}$)". For each x, one of A($\boldsymbol{x}$) and $\neg$A($\boldsymbol{x}$) is provable. The assumption of correctness that Gödel made is that for no formula A(x) and natural-number variable x are there proofs in *S* of all of A(0), A(1), A(2), ..., A(**x**), ... and also of $\neg\forall$xA(x). This assumption he called ω*-consistency.* (*Simple*) *consistency* is the property that for no formula A are there proofs of both of A and $\neg$A. By applying ω-consistency to $\forall$xA, where x is a variable not occurring free in A, ω-consistency implies simple consistency. Restating Gödel's theorem with this terminology: *If S is ω-consistent, it is* (*simply*) *incomplete, that is, there is a closed formula* G *such that neither* G *nor* $\neg$G *is provable in S* (*but* G *is true*).

How could this be? The fundamental fact is that in working with a formal system (apart from its interpretation), the objects we are dealing with (the symbols from a finite or countably infinite collection, the finite sequences of (occurrences) of those symbols, and the finite sequences of such finite sequences) form a countably infinite collection of linguistic objects. By pairing them one-to-one with the natural numbers, or using some other method of associating distinct natural numbers with them (as indeed Gödel did), each object of the formal system is represented by a number, now called its *Gödel number.* So indeed, since the formal system *S* is adequate for a certain part of the elementary theory of the natural numbers, we can express in *S* propositions that by the Gödel numbers actually say things about the system *S* itself. Now Gödel ingeniously constructed his G to be of the form $\forall$xA(x), where A(x) expresses "*x* is not the Gödel number of a proof of the formula with a certain fixed Gödel number *p*" and *p* is the Gödel number of the formula G itself! Thus G says, "Every *x* is not the Gödel number of a proof of me", or simply "I am unprovable." So, if G were provable, G would be false. So (assuming correctness), G is unprovable; hence (by what G says) G is true; and hence (assuming correctness) $\neg$G is also unprovable. It is easy to confirm that ω-consistency suffices as the correctness assumption in concluding that $\neg$G is unprovable, and simple consistency in concluding that G is unprovable.

Gödel's formula G, which says "I am unprovable", is an adaptation of the ancient paradox of *the liar.* The Cretan Epimenides (sixth century B.C.) is reported to have said "Cretans are always liars." If this were the only thing Epimenides said, could it be true? Or false? To take the version of Eubulides (fourth century B.C.), suppose a person says "The statement I am now making is false." If this statement is false, by what it says it would be true; and vice versa. Gödel's substitution of "unprovable" for "false" escapes the paradox, because a statement and its negation can both be unprovable (while they cannot both be false).

In two respects, Gödel's theorem, as given in (1931a), has been improved. Rosser (1936), by using a slightly more complicated formula than Gödel's G, replaced Gödel's hypothesis of ω-consistency by the hypothesis of simple consistency. The other improvement, to be explained next, is connected with a development that took place essentially independently of Gödel's (1931a) and is equally significant.

At least since Euclid in the fourth century B.C., mathematicians have recognized that for some functions and predicates they have "algorithms". An *algorithm* is a procedure described in advance such that, whenever a value is chosen for the variable or a respective value for each of the variables of the function (or predicate), the procedure will apply and enable one in finitely many steps to find the corresponding value of the function (or to decide the truth or falsity of the corresponding value of the predicate).

In the 1930s, the general concept of an "algorithm" came under scrutiny, and Church in (1936) proposed his famous thesis ("Church's thesis" or "the Church-Turing thesis"). This states that all the functions of natural number variables for which there are "algorithms", or which in Church's phraseology are "effectively calculable", belong to a certain class of such functions for which two equivalent exact descriptions had been formulated during 1932–34. Turing in (1937), independently of Church, arrived at the same conclusion, using a third equivalent formulation, namely the functions computable by an idealized computing machine (error-free and with no bound on the quantity of its storage or "memory") of a certain kind, now called a "Turing machine." The thesis applies to predicates, because a predicate can be represented by the function taking 0 as its value when the value of the predicate is true and 1 when it is false.

As Turing wrote (1937, 230): "conclusions are reached which are superficially similar to those of Gödel [in (1931a)]." Gödel (1931a) showed the existence in certain formal systems *S* of "formally undecidable propositions", that is, propositions for which the system *S* does not decide the truth or falsity by producing a proof of A or of $\neg$A, where A is the formula expressing the proposition in the symbolism of *S*. Church (1936) and Turing (1937) showed the existence of "intuitively undecidable predicates", that is, predicates for which there is no "decision procedure" or "effective process" or "algorithm" by which, for each choice of a value of its variable, we can decide whether the resulting proposition is true or false.

In (1936, 1943, 1952), I established a connection between the two developments. The fundamental purpose of using formal systems (as a refinement of the axiomatic-deductive method that has come down to us from Pythagoras and Euclid in the sixth and fourth centuries B.C.) is to remove all uncertainty about what propositions hold in a given mathematical theory. For a formal system to serve this purpose, there must be an algorithm by which we can recognize when we have before us a proof in the system. Furthermore, for the system to serve as a formalization of a given theory, we must have, for the propositions we are interested in, an algorithm to identify the formulas in the system that express those propositions. Of course, we can start with the formulas of the system, if they have an understood interpretation, and take as our class of propositions those expressed by the formulas. For the formalization of the theory of the natural numbers, using Gödel numberings of the formulas and proofs, the algorithms can be for number-theoretic functions and predicates, so the Church-Turing thesis can be applied.

Gödel established his theorem for "*Principia Mathematica* and related systems". In my generalized versions of the theorem, I left out all the finer details of the formalization, and simply assumed that the purpose of formalization as described above is served, for the theory of the natural numbers. Moreover, I chose in advance a fixed number-theoretic predicate $P(a)$ so that every correct formal system fails to formalize its theory completely. Gödel's undecidable propositions in various formal systems are then all values of this one predicate. Thus: *There is a predicate* $P(a)$ *of elementary number theory with the following property. Suppose that in a formal system* S (i) *there are formulas* A_a *for* $a = 0, 1, 2, \ldots$ *given by an algorithm* (which formulas we take to express the propositions $P(a)$ for $a = 0, 1, 2, \ldots$) *such that, for each* $a = 0, 1, 2, \ldots$, A_a *is provable in* S *only if* $P(a)$ *is true, and* (ii) *there is an algorithm for determining whether a given sequence of formulas in* S *is a proof in* S *of a given formula. Then there is a number* p *such that* $P(p)$ *is true but* A_p *is unprovable in* S. *If moreover* (iii) *there are formulas* $\neg A_a$ *such that, for each* $a = 0, 1, 2, \ldots$, $\neg A_a$ *is provable in* S *only if* $P(a)$ *is false, then* $\neg A_p$ *is also unprovable in* S (*so* A_p *is undecidable in* S).

The predicate $P(a)$ can be of a very simple form (suggested in Kleene (1936, Footnote 22), and used in his (1943, 1952): "for all x, $Q(a,x)$" where Q is a decidable predicate. (A fuller exposition is in Kleene 1976, 768–69.)

As I expressed the generalized Gödel theorem in a lecture at the University of Wisconsin in the fall of 1935 (with my (1936) already written, and knowing the contents of Church (1936) but not yet of Turing (1937)), the theory of the natural numbers—indeed just the theory of the limited part of it represented by the predicate $P(a)$—offers inexhaustible scope for mathematical ingenuity. No one will ever succeed in writing down explicitly a list of principles (given as a formal system) sufficient for providing a proof of each of the propositions $P(a)$ for $a = 0, 1, 2, \ldots$ that is true.

To recapitulate, by Gödel's first incompleteness theorem, as he gave it in (1931a), none of the familiar formal systems (like that of *Principia Mathematica*), and by the generalized version of the theorem, which Gödel accepted in a "Note added 28 August 1963" to the van Heijenoort (1967) translation of his (1931a) and in the "Postscriptum" to the Davis (1965) reprint of his (1934), no conceivable formal system, can be both correct and complete for the elementary theory of the natural numbers.

In Gödel's first incompleteness theorem (as stated above for *Principia Mathematica* and related systems), the unprovability of G follows from the assumption that S is simply consistent. By Gödel's numbering, the property of the simple consistency of S can be expressed in S itself by a formula, call it "Consis". And in fact the reasoning by which Gödel showed that "Simple consistency implies G is unprovable" can be formalized within S as a proof of the formula

$$\text{Consis} \rightarrow \text{G},$$

noting that G says "G is unprovable"! Therefore, if Consis were provable in S, by one application of the rule of inference shown first above, G would be, contradicting Gödel's first incompleteness theorem if S is consistent. So we have Gödel's second incompleteness theorem of (1931a): *If S is simply consistent, the formula* Consis *expressing that fact is unprovable in S.*

Hilbert's idea had been to prove the consistency of a suitable formal system *S* of mathematics by finitary methods. In the interesting case that *S* is a formalization embracing some ideal (non-finitary) mathematics, the methods to be used in proving its consistency should not include all those formalized in *S*. Gödel's second theorem shows that not even all the methods formalized in *S* would suffice!

The consequence is that, if Hilbert's idea can be carried out, it cannot be done as simply as presumably had been hoped. Methods will have to be accepted as finitary, and used in the consistency proof of a system *S*, that are not formalizable in *S*. Indeed, this has now been done for the arithmetic of the natural numbers by Gentzen (1936), Ackermann (1940), and Gödel (1958a), and for analysis (real-number theory) by Spector (1961), extending Gödel (1958a).

With the two incompleteness theorems of Gödel (1931a), the whole aspect of work on the foundations of mathematics was profoundly altered.

We have described Gödel's celebrated results in (1930a) and (1931a) in the context of the outlook on foundations at the time. He clearly addressed—and solved—problems existing at that time. Each of these results can be construed as a piece of exact mathematics: on the level of classical number theory in the case of (1930a), and of finitary number-theory in the case of (1931a). The picture mathematicians could entertain of the possibilities for the use of formal systems has been refocused by Gödel's discoveries. Now we know that their use cannot give a resolution of the foundational problems of mathematics as simply as had been hoped. But, in my view, formal systems will not go away as a concern of mathematicians. Recourse to the axiomatic-deductive method, as refined in modern times to formal systems, provides mathematicians with the means of being fully explicit about what they are doing, about exactly what assumptions they have used in a given theory. It is important to have this explicitness when they are engaged in conceiving new methods (as Gödel's first (1931a) theorem shows that for progress they must) and attempting to assure themselves of their soundness (which by Gödel's second theorem cannot be done simply by the metamathematical applications of only the same methods).

In the period we are reviewing (after (1930a) and prior to 1938a)), Gödel made a number of other significant contributions.

In (1934), building on a suggestion of Herbrand (see van Heijenoort (1967, 619)), he introduced the notion of "general recursive functions", which I studied in (1936). This is one of the two equivalent notions that were identified with "effective calculability" by Church's thesis mentioned above. Nevertheless, Gödel did not accept the thesis until later (Kleene 1981, 59–62). Concerning those notions and the third one of Turing, and generalizations of them, a very extensive mathematical theory has been developed (with an important role played by the Herbrand-Gödel notion) and applied to other branches of mathematics (Kleene (1981, 62–64)).

In (1931b), Gödel explained how some of his undecidable propositions become decidable with the addition of higher types of variables, while of course other undecidable propositions can be described. In a trenchant paper (1935), he showed that in the systems with higher types of variables infinitely many of the previously provable formulas acquire very much shorter proofs. He also offered contributions to the so-called *Entscheidungsproblem* (decision problem) for the first-order predicate logic (1930f, 1933b). This is the problem of finding an algorithm, at least for a described class of formulas, for deciding whether a formula is or is not provable. (1931c) is historic as one of the first results on a "formal system" with uncountably many symbols.

The intuitionistic school under Brouwer came to recognize the advantages of formalization for making explicit the boundaries of a given body of theory. So Heyting in (1930a, 1930b) gave a formalization of the intuitionistic logic and of a portion of the intuitionistic mathematics. This has had various mathematical applications. Gödel's papers (1932c, 1932d, 1932e) were important contributions to the study of these systems.

The article of Smorynski and that of Paris and Harrington in Barwise (1977), and Dawson (1979), can serve as a sampling of the reverberations after nearly fifty years from Gödel's (1931a) incompleteness theorems.

GÖDEL'S RELATIVE CONSISTENCY PROOF FOR THE AXIOM OF CHOICE AND FOR THE GENERALIZED CONTINUUM HYPOTHESIS (1938a, 1938b, 1939a, 1939b)

As remarked above, in Cantor's set theory, the set of the sets of natural numbers, and the set of the real numbers, have a cardinal number $2^{\aleph_0}$ greater than the cardinal number $\aleph_0$ of the set of the natural numbers, which is the least infinite cardinal.

In Cantor's theory, the cardinal number $\aleph_1$ next greater than $\aleph_0$ is identified as the cardinal of the set of all possible linear orderings of the natural numbers in which each subset of them has a first member in the ordering ("well-orderings"). Cantor's set theory would be greatly simplified if $2^{\aleph_0}$, which is the infinite cardinal greater than $\aleph_0$ coming to mind first, is actually the next greater cardinal. Cantor conjectured in (1878) that it is, that $2^{\aleph_0} = \aleph_1$. This conjecture is called the "continuum hypothesis" (CH); and it became the central problem of set theory to confirm or refute CH. Sixty years later, with the problem still unsolved, Gödel's results in (1938a, 1938b, 1939a, 1939b) put the matter in a new light.

Using Cantor's *ordinal numbers,* all the infinite cardinals can be listed in order of magnitude as

$$\aleph_0, \aleph_1, \aleph_2, \ldots, \aleph_\alpha, \ldots ,$$

where α ranges over the natural numbers as finite ordinals, and then on into Cantor's infinite ("transfinite") ordinals. The "generalized continuum hypothesis" (GCH) is that, for each ordinal α, $2^{\aleph_\alpha} = \aleph_{\alpha+1}$, where $2^{\aleph_\alpha}$ is the cardinal of the set of all the subsets of a set of cardinal $\aleph_\alpha$.

The theory of sets was axiomatized after the paradoxes had appeared. This consisted in listing a collection of axioms, regarded as true propositions about sets, including axioms providing for the existence of many sets but not of too "wild" sets such as had given rise to the paradoxes. (We recall the Skolem paradox about such systems of axioms in first-order logic.) As a standard list of axioms for set theory, I will take those commonly called the Zermelo-Fraenkel axioms. These arise from the first axiomatization by Zermelo in (1908) by using a refinement proposed by Fraenkel in (1922). One of the axioms, called the "axiom of choice" (AC), has been regarded as less natural than the others. One form of it says that, if we have a collection S of non-empty sets, no two of which have a member in common, there is a "choice set" C containing exactly one member from each set in the collection S. By ZFC I shall mean all the Zermelo-Fraenkel axioms, and by ZF the system of those axioms without AC.

Cantor had not been thinking of his conjecture that $2^{\aleph_0} = \aleph_1$ (CH) relative to a set of axioms. But after choosing an axiomatization, say ZF, there are three possibilities: (1) $2^{\aleph_0} = \aleph_1$ is provable (using elementary logic) from the axioms. (2) $\neg(2^{\aleph_0} = \aleph_1)$ is provable from the axioms. (3) Neither (1) nor (2). This is assuming the axioms are consistent, so that not: (4) Both (1) and (2).

What Gödel did was to exclude (2); he showed that adding

$2^{\aleph_0} = \aleph_1$ to the axioms will not lead to a contradiction (if a contradiction is not already deducible from the axioms without the addition).

To put the matter in its simplest terms, Gödel, using only things about sets justified by the axioms ZF, defined a class L of sets, which he called the "constructible sets", such that all the axioms are true when the "sets" for them are taken to be just the constructible sets L. In effect, L constitutes a kind of skeletal model of set theory—not all the sets presumably intended, but still enough to make all the axioms true. And in this model, AC and CH, and indeed GCH, are all true.

Since nothing is used about sets in this reasoning with L that cannot be based on the axioms ZF, it can be converted as follows into a demonstration that if ZF (taken as the formal system with the mathematical axioms of ZF and the logical axioms and rules of inference of the first-order predicate calculus) is (simply) consistent, so is ZF + AC + GCH (similarly taken). Suppose ZF is consistent, and (contrary to what we want to prove) that a pair of contradictory formulas A and $\neg$A (which we can take to be closed) are provable in ZF + AC + GCH. Let B^L come from any closed formula B by replacing each part of the form $\forall$xC by $\forall x(x \varepsilon L \rightarrow C)$ and each part of the form $\exists$xC by $\exists$x(xεL & C), in effect restricting the variable x to range over L. Here xεL is definable within ZF. Now for the axioms A_0, A_1, A_2, ... of ZF + AC + GCH, we can prove in ZF A_0^L, A_1^L, A_2^L, ..., and then continue by the reasoning that gave the contradiction A and $\neg$A in ZF + AC + GCH to get the contradiction A^L and $\neg A^L$ in ZF, contradicting our supposition that ZF is consistent. Thus Gödel gave a consistency proof for ZF + AC + GCH relative to ZF.

It is natural to ask whether one can also rule out (1), that is, whether the negation $\neg 2^{\aleph_0} = \aleph_1$ of the continuum hypothesis can be added consistently to ZF or indeed to ZFC (provided ZF is consistent). It remained for Paul Cohen in (1963, 1964) to do this. He accomplished this by using another model (quite different from Gödel's) in which all the axioms of ZFC and also $\neg 2^{\aleph_0} = \aleph_1$ hold. (For a comment by Gödel, see 1967.)

Thus, combining Gödel's and Cohen's results, $2^{\aleph_0} = \aleph_1$ is independent of ZFC. Similarly, combining results of Gödel and Cohen, AC is independent of ZF. These results of Gödel and Cohen have ushered in a whole new era of set theory, in which a host of problems of the consistency or independence of various conjectures relative to this or that set of axioms are being investigated by constructing models.

PRINCETON (IAS) 1939–1978

After Gödel's return to Princeton in 1939, he never again left the United States. He became a U.S. citizen in 1948. He received annual visiting appointments from the Institute for Advanced Study from 1940–41 on, became a permanent member in 1947, a professor (in the School of Mathematics) in 1953, and retired in 1976. He was keenly interested in the affairs of the Institute, and conscientious in work for the Institute, especially in the evaluation of applicants.

I have already reviewed most of his work that was published in his own papers. The last paper of his in the Bibliography (1958a), mentioned above, gives a new interpretation of intuitionistic number theory, which Wang (1981, 657) says "was obtained in 1942. Shortly afterwards he lectured on these results at Princeton and Yale." Gödel's (1944, 1947) are exceedingly suggestive expository and critical articles on Russell's mathematical logic and on the continuum problem.

In December 1946, Gödel presented a paper to the Princeton Bicentennial Conference on Problems of Mathematics, published in 1965, suggesting a non-constructive extension of formal systems, or of the notion of "demonstrability", to be obtained by using stronger and stronger "axioms of infinity" asserting the existence of large cardinal numbers in set theory. He wrote, "It is not impossible that for such a concept of demonstrability some completeness theorem would hold which would say that every proposition expressible in set theory is decidable from the present axioms plus some true assertion about the largeness of the universe of all sets." The paper makes a similar suggestion regarding the concept of mathematical "definability".

Gödel and Einstein, both at the Institute for Advanced Study, saw much of each other. Because of Gödel's interest in Kant's philosophy of space and time, Gödel became interested in general relativity theory, on which he worked during 1947 to 1950 or 1951. Three short articles (1949a, 1949b, 1950) resulted. According to R. Penrose, as reported in Kreisel (1980, 214–15), "[these articles] were highly original and, in the long run, quite influential. . . . Gödel's work served as a cross check on mathematical conjectures and proofs in the modern global theory of relativity." (For summaries, see Kreisel loc. cit. and Dawson 1983, 266–67, the Addenda and Corrigenda to which reports on a controversy about it.)

Gödel was deeply interested in philosophy, and in the relevance of philosophical views to the mathematical problems with which his work dealt. Wang writes (1981, Footnote 9), "we may conjecture that between 1943 and 1947 a transition occurred from Gödel's concentration on mathematical logic to other theoretical interests which are primarily philosophical. . . . From [his papers (1946, 1947)] one gets the clear impression that Gödel was interested only in really basic advances." Kreisel (1980, 204–13) calls Gödel's first proposal in (1946) "Gödel's programme", and discusses it while citing

Kanamori and Magidor (1977) for more complete references to the work done on the program over the last thirty-five years.

According to Wang (1978, 183; 1981, 658), Gödel worked on several papers (as early as 1947, perhaps), which in the end he did not publish. One of these was his Josiah Willard Gibbs Lecture, *Some Basic Theorems on the Foundations of Mathematics and their Philosophical Implications,* which he read from a manuscript to the American Mathematical Society on December 26, 1951 (I was present). Some of the ideas in this lecture are reported by Wang in the pages cited as Gödel (1974a). Gödel left a considerable quantity of notes (almost 5,000 pages, according to Kreisel 1980, 151). Undoubtedly, these will constitute a mine for scholars for quite some time into the future.

Gödel contributed reflections on some of his papers as emendations, amplifications, and additions to reprints and translations (see Bibliography, 1939, 1946, 1949b, 1963–66a).

Gödel was rather retiring. But he was kind and responsive to qualified interlocutors who took the initiative to engage him in discussions. So it has come about that various reflections and views of his have been reported with his permission in writings by other authors. Also, on occasions, he took the initiative to volunteer pronouncements other than in papers of the usual sort. I have included in this Bibliography all the material of these two kinds that has come to my attention (without attempting to draw a line between the substantial, and the slight). This accounts for (1931d) and all of the items after (1950), except (1958a) and (1963–66a).

On the occasion of the award of an Einstein Medal to Gödel on March 14, 1951, John von Neumann began his tribute to Gödel (von Neumann 1951) with the words:

Kurt Gödel's achievement in modern logic is singular and monumental—indeed it is more than a monument, it is a landmark which will remain visible far in space and time.

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HONORS AND DISTINCTIONS

AWARDS AND MEMBERSHIPS

Albert Einstein Award, Lewis and Rosa Strauss Memorial Fund (shared with Julian Schwinger), 1951
National Academy of Sciences, Member, 1955
American Academy of Arts and Sciences, Fellow, 1957
American Philosophical Society, Member, 1961
London Mathematical Society, Honorary Member, 1967
Royal Society (London), Foreign Member, 1968
British Academy, Corresponding Fellow, 1972
Institut de France, Corresponding Member, 1972
Académie des Sciences Morales et Politiques, Corresponding Member, 1972
National Medal of Science, 1975

HONORARY DEGREES

D.Litt., Yale University, 1951
Sc.D., Harvard University, 1952
Sc.D., Amherst College, 1967
Sc.D., Rockefeller University, 1972

ANNOTATED BIBLIOGRAPHY

I have listed the original items in the order of their authorship by Gödel, insofar as I could find information to base this on. Thus 1939b, communicated by Gödel on February 14, 1939, is put after 1939a, which is a set of notes by George W. Brown published in 1940 on lectures delivered by Gödel in the fall term of 1938–39; and 1934 and 1946, only published in 1965, are in the right order. This has involved using dates of presentation for the eleven items listed from *Ergebnisse eines mathematischen Kolloquiums* (ed. Karl Menger). Not listed are four brief contributions by Gödel to discussions in these *Ergebnisse* (4:4, 4:6, 4:34(51), 7:6), twenty-seven reviews by Gödel in *Zentralblatt für Mathematik und ihrer Grenzgebiete* 1931–36, six reviews by Gödel in *Monatshefte für Mathematik und Physik* 1931–33, and the Spanish translations in Mosterín (1981) of all of the papers of Gödel except 1932a, 1932b, and 1933a. (In Dawson (1983), the four *Ergebnisse* items are cited (just before [1933a] and as [1933], [1933d], [1936]), as well as the twenty-seven *Zentralblatt* reviews and the Mosterín (1981) translations, while the six *Monatshefte* reviews are cited in its Addenda and Corrigenda.)

The other translations and reprintings of Gödel's papers are cited within the items for the originals. Eight of the translations (with no inputs by Gödel, as far as I know) are thus cited concisely through their reviews or listings in the *Journal of Symbolic Logic*. Books (not journals) are generally cited through the Reference section. For example, the English translation of 1930a is on pp. 583–91 of the book listed in the References as "van Heijenoort, J., ed. 1967."

A work entitled *Kurt Gödel, Sein Leben und Wirken,* W. Schimanovich and P. Weibel, eds., is to be published by Verlag Hölder-Pichler-Tempsky, Vienna. It will contain some of Gödel's works and various biographical and interpretative essays.

The Association for Symbolic Logic is arranging for the publication in the original (by Oxford University Press, ed. by S. Feferman et al.), and when the original is in German also in English translation, of all of Gödel's published works, with introductory historical notes to them and a biographical introduction and survey. (Volume I (1986) contains Gödel's published works up through 1936; the rest will be in Volume II, probably in 1986. A further volume or volumes are projected to contain a selection of unpublished material from Gödel's *Nachlass.*)

1930

a. Die Vollständigkeit der Axiome des logischen Funktionenkalküls. Monatsh. Math. Phys., 37:349–60. (English trans: van Heijenoort, 1967, pp. 583–91, with two comments by Gödel, pp.

510–11; also see Kleene, 1978; reprinted in: Berka and Kreiser, 1971, pp. 283–94.)

b. Über die Vollständigkeit des Logikkalküls (talk of 6 Sept. 1930). Die Naturwissenschaften, 18:1068.

c. [Remarks in] Diskussion zur Grundlegung der Mathematik [7 Sept. 1930]. Erkenntnis (1931), 2:147–48.

d. Nachtrag [to the preceding remarks]. Erkenntnis, 2:149–51. (Italian trans: Casari, 1973, pp. 55–57.)

e. Einige metamathematische Resultate über Entscheidungsdefinitheit und Widerspruchsfreiheit. Anz. Akad. Wiss. Wien, Math.-naturwiss. Kl. 67:214–15. (English trans.: van Heijenoort, 1967, pp. 595–96; reprinted in: Berka and Kreiser, 1971, pp. 320–21.)

f. Ein Spezialfall des Entscheidungsproblems der theoretischen Logik. Ergeb. math. Kolloq. (for 1929–30, publ. 1932), 2:27–28.

1931

a. Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme I. Monatsh. Math. Phys., 38:173–98. (English trans., 1962, see J. Symb. Logic, 30:359–62; also in: Davis, 1965, pp. 5–38 (see J. Symb. Logic, 31:486–89); with a note by Gödel, in: van Heijenoort, 1967, pp. 596–616. Italian trans.: Agazzi, 1961, pp. 203–28; Portuguese trans.: Lourenço, 1979, pp. 245–90.)

b. Über Vollständigkeit und Widerspruchsfreiheit. Ergeb. math. Kolloq. (for 22 Jan. 1931, publ. 1932), 3:12–13. (English trans., with a remark by Gödel added to Ftn. 1: van Heijenoort, 1967, pp. 616–17.)

c. Eine Eigenschaft der Realisierung des Aussagenkalküls. Ergeb. math. Kolloq. (for 24 June 1931, publ. 1932), 3:20–21.

d. Letter to Zermelo, October 12, 1931. In: Grattan-Guinness, 1979, pp. 294–304.

e. Über Unabhängigkeitsbeweise im Aussagenkalkül. Ergeb. math. Kolloq. (for 2 Dec. 1931, publ. 1933), 4:9–10.

1932

a. Über die metrische Einbettbarkeit der Quadrupel des R_3 in Kugelflächen. Ergeb. math. Kolloq. (for 18 Feb. 1932, publ. 1933), 4:16–17.

b. Über die Waldsche Axiomatik des Zwischenbegriffes. Ergeb. math. Kolloq. (for 18 Feb. 1932, publ. 1933), 4:17–18.

c. Zum intuitionistischen Aussagenkalkül. Anz. Akad. Wiss. Wien, Math.-naturwiss. Kl. (for 25 Feb. 1932), 69:65–66. (Reprinted, with an opening clause attributing the question to Hahn, in Ergeb. math. Kolloq. (for 1931–32, publ. 1933), 4:40; and in Berka and Kreiser, 1971, p. 186.)

d. Zur intuitionistischen Arithmetik und Zahlentheorie. Ergeb. math. Kolloq. (for 28 June 1932, publ. 1933), 4:34–38. (English trans.: Davis, 1965, pp. 75–81 (see J. Symb. Logic, 31:490–91). Portuguese trans.: Lourenço, 1979, pp. 359–69.)

e. Eine Interpretation des intuitionistischen Aussagenkalküls. Ergeb. math. Kolloq. (for 1931–32, publ. 1934), 4:39–40. (English trans., 1969, see J. Symb. Logic, 40:498; reprinted in: Berka and Kreiser, 1971, pp. 187–88).

f. Bemerkung über projektive Abbildungen. Ergeb. math. Kolloq. (for 10 Nov. 1932, publ. 1934), 5:1.

1933

a. With K. Menger and A. Wald. Diskussion über koordinatenlose Differentialgeometrie. Ergeb. math. Kolloq. (for 17 May 1933, publ. 1934), 5:25–26.

b. Zum Entscheidungsproblem des logischen Funktionenkalküls. Monatsh. Math. Phys. (received 22 June 1933), 40:433–43. (For a correction, see Goldfarb, 1981.)

1934

On Undecidable Propositions of Formal Mathematical Systems. Mimeographed notes by S. C. Kleene and J. B. Rosser on lectures at the Institute for Advanced Study, Feb.–May, 1934, 30 pp. (Extensively distributed, deposited in some libraries, and listed in the J. Symb. Logic Bibliography 1:206; printed with corrections, emendations, and a Postscriptum, by Gödel in Davis 1965, pp. 41–74 (see J. Symb. Logic, 31:489–90). A relevant Gödel letter of 15 Feb. 1965 is quoted there on p. 40, and in Kleene, 1981, pp. 60, 62, and of 23 April 1963 in van Heijenoort 1967, p. 619. Portuguese trans. in Lourenço 1979, pp. 291–358.)

1935

Über die Länge von Beweisen. Ergeb. math. Kolloq. (for 19 June 1935, with a remark added in the printing 1936), 7:23–24. (English trans.: Davis 1965, pp. 82–83 (see J. Symb. Logic. 31:491). Portuguese trans.: Lourenço 1979, pp. 371–75.)

1938

a. The consistency of the axiom of choice and of the generalized continuum-hypothesis. Proc. Natl. Acad. Sci. USA (communicated 9 Nov. 1938), 24:556–57.
b. The consistency of the generalized continuum-hypothesis. Bull. Am. Math. Soc. (abstract of a talk on 28 Dec. 1938, publ. 1939), 45:93.

1939

a. *The Consistency of the Axiom of Choice and of the Generalized Continuum-Hypothesis with the Axioms of Set Theory.* Notes by G. W. Brown on lectures at the Institute for Advanced Study during the fall term of 1938–39. Ann. Math. Stud., no. 3. Princeton, N.J.: Princeton U. Press, 1940. (Reprinted 1951 with corrections and three pages of notes by Gödel; the seventh and eighth printings, 1966 and 1970, include additional notes and a bibliography. Russian trans. 1948, see J. Symb. Logic. 14:142.)
b. Consistency-proof for the generalized continuum-hypothesis. Proc. Natl. Acad. Sci. USA (communicated 14 Feb. 1939), 25:220–24. (Reprinted in Felgner, 1979; corrections in (1947, Ftn. 23, = Ftn. 24 in the 1964 reprint); also see Kleene 1978 and Wang 1981, Ftn. 7.)

1944

Russell's mathematical logic. In Schilpp (1944, pp. 123–53). (Reprinted, with a prefatory note by Gödel, in Benacerraf and Putnam 1964, pp. 211–32; Italian trans.: 1967, see J. Symb. Logic, 34:313; French trans.: 1969, see J. Symb. Logic, 40:281; reprinted, with Gödel's 1964 prefatory note expanded, a reference supplied in Ftn. 7, and Ftn. 50 omitted, in Pears 1972, pp. 192–226; Portuguese trans.: Lourenço 1979, pp. 183–216.)

1946

Remarks before the Princeton Bicentennial Conference on Problems in Mathematics [December 17], 1946. Plans for publication of the papers presented at the conference fell through, as the conferees learned only much later. When the Davis anthology

(1965) was being planned, Kleene drew the attention of the publisher to this paper of Gödel and supplied a copy of the text that had been in his file since 1946, which with Gödel's permission (and Gödel's addition of a four-line footnote) was then published as Davis (1965, 84–88). (Italian trans.: 1967, see J. Symb. Logic, 34:313; reprinted, with trifling changes in punctuation and phrasing, and the substitution of "It follows from the axiom of replacement" for "It can be proved" at the end, in Klibansky 1968, pp. 250–53; Portuguese trans.: Lourenco 1979, pp. 377–83.)

1947

What is Cantor's continuum problem? Am. Math. Mon., 54:515–25; errata, 55:151. (Reprinted, with some revisions, a substantial supplement, and a postscript, by Gödel, in Benacerraf and Putnam 1964, pp. 258 -73; Italian trans.: 1967, see J. Symb. Logic, 34:313; Romanian trans.: Pârvu 1974, pp. 317–38; Portuguese trans.: Lourenço 1979, pp. 217–44; see (1958c) and (1973).)

1949

a. An example of a new type of cosmological solutions of Einstein's field equations of gravitation. Rev. Mod. Phys., 21:447–50.
b. A remark about the relationship between relativity theory and idealistic philosophy. In Schilpp (1949, pp. 555–62). (German trans., with some additions by Gödel to the footnotes: Schilpp 1955, pp. 406–12.)

1950

Rotating universes in general relativity theory. In: *Proc. Int. Cong. Math. (Cambridge, Mass., 1950)*, vol. 1, pp. 175–81. Providence, R.I.: American Mathematical Society, 1952.

1952

[A popular interview with Gödel:] Inexhaustible. The New Yorker, Aug. 23, 1952, pp. 13–15.

1956

Gödel expresses regret at Friedberg's intention to study medicine in: The prodigies. Time, March 19, 1956, p. 83.

1957

Kreisel (1962, pp. 140–42) states some results as having been communicated to him by Gödel in 1957.

1958

a. Über eine bisher noch nicht benützte Erweiterung des finiten Standpunktes. Dialectica, 12:280–87. (Reprinted in *Logica, Studia Paul Bernays Dedicata,* Bibliotheque Scientifique no. 24, pp. 76–83. Neuchatel: Griffon, 1959. Russian trans. 1967, see J. Symb. Logic, 35:323; English trans.: 1980 [with a bibliography of work resulting from this paper], J. Philos. Logic, 9:133–42. According to the review of this translation by Feferman, Math. Rev., 81i:3410–11, there was an unpublished earlier English trans., which was revised several times by Gödel and "contained a number of further notes which considerably amplified and in some cases corrected both technical and philosophical points." Italian trans.: Cagnoni 1981, pp. 117–23; also see (1961) and Spector (1961).)
b. Kreisel (1958, pp. 321–22) attributes the substance of his remarks 2.1 and 2.3 to Gödel.
c. A statement by Gödel is quoted in Ulam (1958, Ftn. 5, p. 13).

1961

A postscript by Gödel is on p. 27 of Spector (1961).

1963–66

a. Benacerraf and Putnam (1964), Davis (1965), and van Heijenoort (1967) include various contributions by Gödel to their reprints and translations of his (1930a), (1931a), (1931b), (1934), (1944), and (1947).
b. A letter from Gödel is quoted in von Neumann (1966, pp. 55–56).
c. A 1966 greeting by Gödel is on p. (viii) of Bulloff et al. (1969).

1967

An extract from a 30 June 1967 letter from Gödel is in Rautenberg (1968, p. 20).

1973

A communication from Gödel of October 1973 is quoted in Greenberg (1980, p. 250).

1974

a. Communications from Gödel are reproduced in Wang (1974, pp. 8–12, 84–88, 186–90, and 324–26).
b. Gödel contributed a statement to the preface to Robinson (1974).
c. Reinhardt (1974, Ftn. 1, p. 189) reports on discussions with Gödel.
d. A 1974 memorial tribute to Robinson by Gödel appears opposite the frontispiece of Saracino and Weispfennig (1975).

1976–77

a. Wang (1981), begins with the words, "The text of this article [but not the footnotes and section headings] was done together with Gödel in 1976 to 1977 and was approved by him at that time."
b. The text of Kleene (1978) is composed of communications from Gödel of May and June 1977. See Kreisel's review in the Zentralblatt (1979) 401:12–13.