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EDWARD KASNER

1878—1955

A Biographical Memoir by

JESSE DOUGLAS

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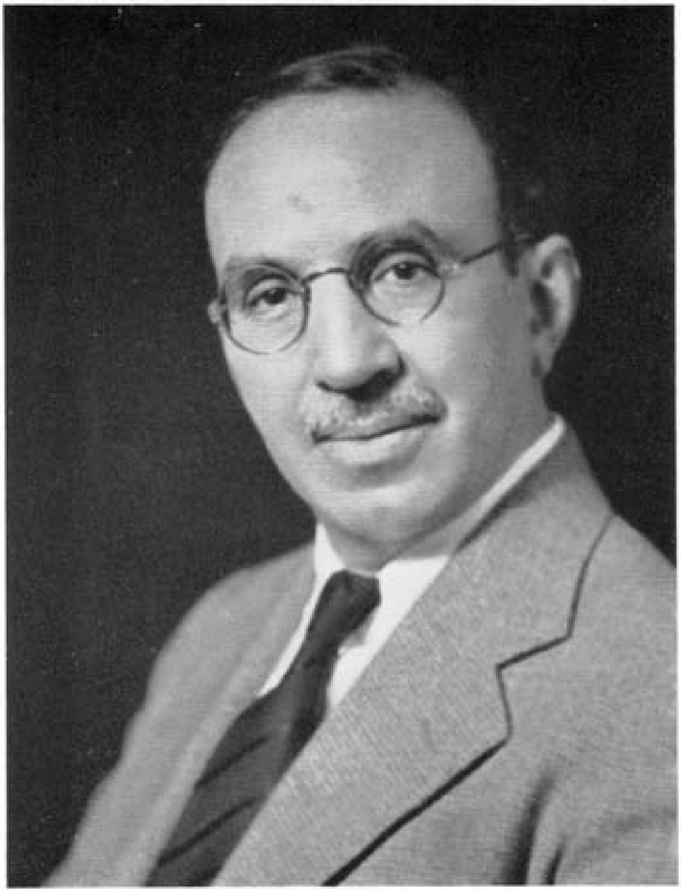
Biographical Memoir



Edward Kasner

EDWARD KASNER

April 2, 1878—January 7, 1955

BY JESSE DOUGLAS

ON JANUARY 7, 1955, the distinguished American mathematician Edward Kasner passed away. To the writer of this biographical memoir he was for forty years teacher, colleague, and friend. The following account is an attempt to do adequate honor to his memory by recalling those features of his life, his personal nature, and his scientific work which seem to me most typical and noteworthy.

LIFE HISTORY

Edward Kasner was born in New York City on April 2, 1878, the son of Bernard and Fanny Kasner. He was the sixth of a family of eight brothers and sisters, for whom his affection and loyalty over the years formed one of the cornerstones of his character.

As a boy, he went to Public School No. 2, near his home in lower Manhattan, where even then he was singled out by his teachers as one of most unusual ability and promise. Kasner himself once said that the first time he ever heard the word "mathematician" was when one of his teachers prophesied that he, Eddie, would some day become a great one. This occurred at the age of eight, on the occasion of his solving an unusually difficult problem in the arithmetic class.

Finding elementary school arithmetic an insufficient challenge to his ability, the boy Kasner obtained an algebra text and would often occupy himself during the arithmetic lesson by solving simultaneous quadratics on his slate—then in vogue as a writing tablet in the New York City public schools.

After graduation from elementary school at the age of thirteen with the highest honors, including the gold medal for general excellence in all studies, Kasner became a student at the College of the City of New York. At that time, this institution offered a five-year course of study, combining what would now be high school and college work, and leading to the degree of Bachelor of Science.

Kasner was a most brilliant student at City College during the years 1891–1896. The versatility of his mind and the wide scope of his intellectual gifts and interests—by no means confined to mathematics—become apparent from a list of the medals awarded to him during his course of study. The subjects included: astronomy, logic, natural philcsophy, and political science, besides the gold and silver medals in mathematics. To any one who knows the intensity of competition for these honors at City College, with its large number of serious and capable students, this achievement of the young Kasner is most impressive.

For his graduate work, Kasner went to Columbia University. He received the Master of Arts degree in 1897, and the degree of Doctor of Philosophy in Mathematics in 1899. The records suggest that his was probably only the second or third doctorate in mathematics awarded by Columbia.

It must be remembered that at this time there was in this country no established living tradition of essentially American mathematics. Kasner followed the custom then prevalent of going to Europe for postgraduate study, spending a year at Göttingen, where he attended chiefly the lectures of Felix Klein and David Hilbert.

Returning to Columbia, Kasner was appointed in 1900 to the position of tutor in mathematics at Barnard College. In 1905 he was made instructor, and in 1906 was promoted to adjunct professor, in which post he served till 1910. In that year he transferred his affiliation to Columbia University proper, where the trustees appointed him to a full professorship. In this capacity he served for the remainder of his career. In 1937 he was appointed to the chair of mathematics founded by Robert Adrain, and in 1949 was retired with the title of Adrain Professor Emeritus.

Many honors came to Kasner during the course of his professional career. He was selected as one of the principal speakers at the International Congress of Arts and Sciences at St. Louis in 1904, and at the Harvard Tercentenary in 1936. He served as vice-president of the American Mathematical Society (1908), and of the American Association for the Advancement of Science (1906); he was also chairman of the section of Mathematics and Astronomy in the latter organization. At various times he was chosen as one of the editors of the *Bulletin* and of the *Transactions* of the American Mathematical Society. His election to membership in the National Academy of Sciences came in 1917.

In July of 1951, Kasner's lifetime record of solid health was shattered by a sudden cerebral stroke. The effect on his morale was devastating; his freedom-loving spirit simply could not tolerate the limitations which his physical condition imposed.

From this state of mind he was lifted briefly by the award to him by Columbia of an honorary doctorate of science; this took place October 31, 1954, at the bicentennial convocation of the University. I remember well Kasner's great pride at receiving this distinction, as part of a most impressive ceremony in the Cathedral of St. John the Divine, near the campus. Another recipient of an honorary degree on the same occasion was Queen Mother Elizabeth of England, then on a visit to this country.

From this time on, however, Kasner continued to decline rapidly in body and spirit, till the end came mercifully during the first week of 1955.

His was a broad, unfettered type of genius, a wide soaring kind of imagination, chafing at restrictions and discipline of any sort—to him, these seemed to impede the way to true insight and development. The kind of outlook and approach to mathematics and science which he exemplified appears today, regrettably, to be rapidly diminishing in practice and influence. We shall not soon again see one with Kasner's special variety of mental gifts.

PERSONAL TRAITS AND PREDILECTIONS

Kasner had two great loves: for nature and for children. The woods adjoining the Palisades on the Hudson shore opposite Columbia were frequently his playground and rendezvous. Here he enjoyed taking long walks with friends or colleagues, conversing with them about any topic, mathematical, philosophical, or political, that might interest him at the time.

I well remember some of those typical Saturday afternoon excursions into the green countryside. We—usually a small group of colleagues—would meet Kasner at the Fort Lee ferry and take the ride to the Jersey shore, then the trolley car to the top of the Palisades. On the way to the woods we might stop to buy some cookies; these were to consume with the tea which Kasner brewed in a pot, dug up from its hiding place under a log, where it had been left at the end of the previous excursion. It was amusing that Kasner could direct us unerringly to this cache in spite of the absence of any markers that we could notice.

Water was obtained by Kasner from a nearby stream, and heated over a fire built from dead branches and leaves which we had gathered. Around this we would then sit through the afternoon, conversing about our teaching at Columbia, about mathematical research, and about random topics of interest. As the sun faded and darkness began to fall, Kasner would arise at the psychological moment signalled by a halt in the conversation, and carefully extinguish the fire with some water kept in reserve—we then knew it was time to start for home.

Besides nature, Kasner's other strong liking was for children. Having none of his own, he directed his affection towards his young nephews and nieces. Uncle Edward taking a walk with a child of one of his brothers or sisters was a familiar sight in the vicinity of the Columbia campus. He would attempt, very informally, to teach them the highest type of mathematics which they seemed capable of appreciating—often indeed some popularized version of a topic upon which he was currently lecturing. It was his belief that children, unspoiled by too much indoctrination in routine courses, were often brighter in seeing into essentials than many adults.

Kasner was frequently invited to lecture on mathematics to children in kindergartens and private schools. One of his favorite topics was the packing of equal circles in a plane. "How many pennies can you place on the floor touching one penny?" would be the opening question of his talk. This choice of subject illustrates Kasner's ability to select from material of an advanced nature some characteristic part sufficiently elementary to interest and to be comprehended by intelligent nonmathematicians. Indeed, the three-dimensional generalization of the penny distribution problem is the very difficult one of the packing of equal spheres. One of the aspects of this problem, the maximum number of nonoverlapping unit spheres which can be placed tangent to a given unit sphere, has been solved only recently (1953) by van der Waerden and Schütte with quite advanced methods. The answer is *twelve,* incidentally, refuting the conjecture thirteen made by David Gregory, a contemporary of Newton.

In his conversational style, and sometimes in his writing, Kasner was often what might be called whimsical. He would indulge in epigrammatic, paradoxical forms of expression, sometimes, one thought, with the purpose of startling his listener into attention, sometimes in order to exercise his own sense of humor. To some this habit might on occasion prove disturbing, until it was noticed that Kasner had his tongue in cheek all the while.

Another of Kasner's interests was art, particularly sketching and painting in water colors. During his frequent trips to Europe he would often attend a school of art in Paris. He was a life member of the Metropolitan Museum of Art, and spent many afternoons, when his teaching duties were over, strolling through the galleries, looking at the most recently arrived works of painting or sculpture or some current archaeological find. As I noticed on some occasions when I accompanied him, his visits were sufficiently frequent to allow him to become well-known personally to the museum employees, and even to the traffic policeman stationed outside.

Another place often visited by Kasner with a young niece or nephew was the Hayden Planetarium. It was his delight to supplement the formal lecture with his own explanation of the wonders of astronomy to the child.

In general, Kasner was a man-about-New York. His relatively moderate teaching schedule gave him the opportunity which he enjoyed of visiting almost every place of historical or topical interest in and near the city, or any spot which offered a pleasant stroll and a nearby eating place where he could sit down for rest and refreshment. Often alone, sometimes with a colleague, relative, or friend, he made frequent visits to places like Central Park, Radio City, Governor's Island, Staten Island, Coney Island, the Palisades of the Hudson, or Bear Mountain State Park.

TEACHER AND LECTURER

Kasner's style of teaching was inspirational and suggestive, rather than systematic and disciplinary; indeed the latter was completely foreign to his temperament and to his own way of thinking. He was certainly no friend of that exaggerated "rigor," which nowadays seems so fashionable in certain quarters. "Rigor is for dull people" was a statement characteristic of his attitude, which I once heard him utter. To him, complete precision in formulation of an idea, or in the idea itself, was a mirage, an illusion, since after all everything had to be conveyed with human words and was subject to the hazards of human interpretation and understanding. By this, I do not mean that he was in any sense careless in his scientific work or its presentation; he simply believed in such a degree of accuracy as was sufficient for the clear comprehension of the theory or problem at hand. Anything more than this he regarded as affectation, positively harmful in its tendency to block the road to that clear insight into essentials, and view of related problems, which he considered paramount.

Kasner's undergraduate teaching at Barnard College might be said to have had for its main object the arousing in his students' minds of an interest in mathematics, the desire to learn more about the subject, and perhaps to do some work in it. On the mathematics bulletin board near his office would be posted samples of his students' work which he considered of special merit or interest. Here might appear, for example, a carefully drawn figure of the "snowflake curve" (a certain continuous curve without a tangent) approximated by a patiently constructed polygon of 3,072 tiny sides. Another exhibit might be the graph of commutative exponentiation: $x^y = y^x$, or the anti-snowflake curve, or the curve obtained by snipping off successively in each corner of a given polygon a triangle whose sides adjacent to this corner were a fixed fraction $r < \frac{1}{2}$ of the sides of the previous polygon. One started with a square P_4 and obtained in this way an infinite sequence of convex polygons: P_4, P_8, P_{16}, P_{32}, . . . , with a definite limiting curve C. The mathematical difficulties involved in the study of the nature of C are by no means trivial: for instance, as shown by Kasner and his students, this curve consists of four conjoined parabolic arcs when $r = \frac{1}{4}$, and for all other values of r is nonanalytic.

It was considered a great honor for any student to have her work posted on "Professor Kasner's bulletin board," and through the years this display of the mathematical work of outstanding students was an established Barnard tradition.

However, the center of Kasner's instruction at Columbia was his "Seminar in Differential Geometry," designed for graduate students and for those seeking to do research in the subject. I think it may truly be said that nearly every mathematician who arose in the New York area during the first half of this century—whether his main interest was differential geometry or not—attended this seminar at some time during his course of study, and derived from it illumination and inspiration. Omitting the present writer, some of the names that come to mind are: Joseph F. Ritt, George A. Pfeiffer, Gabriel Green, Emil L. Post, Philip Franklin, Aaron Fialkow, George Comenetz, John De Cicco, and Don Mittleman. These listeners and disciples carried his influence, direct or indirect, into their teaching and research. Far more doctorates in mathematics were awarded under Kasner's direction than under that of any other professor at Columbia.

Kasner's mode of presentation was ideal for those seeking fruitful research problems. The theme of his advanced teaching, in any mathematical situation under discussion was this: "Have we a problem here? Is there something still incomplete, something of significance or interest left to be investigated? Is there a property stated which is not characteristic?—then find a characteristic property. Have we a condition which is sufficient but not necessary, or vice versa?—then find one which is both. Is the converse of this theorem also true? If so, state and prove it; if not, find a limited converse. Are the hypotheses implying a stated result independent, or are some of them in whole or in part redundant? If so, exhibit the dependence and accordingly prove a stronger theorem." It is surprising how many worth-while directions of research were revealed by this approach.

Kasner's attitude towards his students was most sympathetic; for any accomplishment he always had warm praise and encouragement towards further achievement. Indeed, his enthusiasm and emphatic endorsement of many of his students often had the effect of arousing the skepticism of some of his more cautiously tempered colleagues. But, frequently enough, Kasner turned out to be essentially right in his appraisal.

His penchant for making the essential features of the most advanced mathematics clear and interesting to intelligent laymen caused Kasner to be in demand as a lecturer in several educational institutions in New York and elsewhere in the country. For a number of years his evening course in "Fundamental Concepts of Mathematics" was well-attended and very favorably received by the audience at the New School of Social Studies. His yearly trip to St. John's College at Annapolis, Maryland, on invitation, to deliver a lecture on some mathematical topic that interested him currently, was an event that was awaited with high anticipation by the student body and faculty of that institution of the "hundred great books"—some of which, indeed, Kasner himself had a share in selecting. Dean Scott Buchanan often referred to Kasner as "my favorite mathematical lecturer."

Almost every April, at the Annual Meeting of the National Academy of Sciences, Kasner would present some of his current research in the form of a paper read at one of the scientific sessions. Here again, his expository gifts came to the fore. He knew how to suppress details too technical for his non-mathematical audience, and yet be informative and interesting, leaving everyone with a clear impression of the essential features of his work. His style of presentation was often held up as a model by members of the other sections and by the committees of the Academy appointed to study matters of this kind.

Kasner's most widely known and disseminated popular work on mathematics was his book *Mathematics and the Imagination,* written with his student James Newman in 1940. This was a presentation, designed for the educated layman, of some of those mathematical topics which were always favorites with Kasner: large numbers, crinkly curves, higher dimensional geometry, some surprising facts of topology, etc. Kasner was much gratified by the highly favorable reception this book received from reviewers and readers; for a considerable time it stood high on the list of best sellers. Translations were published in Spanish and Swedish; one in Japanese was also planned.

In connection with the topic of large numbers, Kasner introduced the terms "googol" (10^{100}) and "googolplex" (10^{googol}), invented by one of his young nephews at Kasner's instigation. These names have been incorporated into recent editions of Funk and Wagnall's dictionary as standard words of the English language, and I have heard them used quite casually on the radio.

Kasner kept in continual scientific touch with the great men in his field all over the world. Mathematicians of the stature of Edward Study in Germany and Tullio Levi-Civita in Italy were in the habit of sending him reprints of their publications. Kasner would discuss some of these or have reports delivered upon them in his seminar. My own doctoral dissertation extended and generalized the results of a paper, combining differential geometry and geometric optics, which was sent to Kasner by Levi-Civita.

SCIENTIFIC WORK

Kasner's principal mathematical contributions were in the field of geometry, chiefly differential geometry. His basic mental outlook and equipment, imaginative, intuitive, concrete, visual, fitted him most naturally for work in this branch of mathematics. He observed that many propositions in mathematical physics, in analysis, in algebra, had an essence that was purely geometrical, and could be stated entirely in terms of spatial relations without reference to other concepts peculiar to the special subject, such as force, mass, time, or number. His program was then to distill this geometric essence, and to analyze it from various points of view, particularly with regard to any interesting problems that might be suggested in this way.

A rough subdivision of Kasner's scientific career might be made into four periods according to his dominant interest at the time: Differential-Geometric Aspects of Dynamics (1905–1920), Geometric Aspects of the Einstein Theory of Relativity (1920–1927), Polygenic Functions (1927–1940), Horn Angles (1940–1955).

Preceding this main work was Kasner's doctoral dissertation, "On the Invariants of the Inversion Group," published in the *Transactions of the American Mathematical Society,* 1900. This was written in the spirit of Klein's Erlangen program, according to which geometrical properties are to be classified according to the group of transformations under which these properties are invariant: e.g., metric, affine, projective, inversive, conformal, etc. In reading this paper, even at the present time, one must admire the virtuosity and mastery shown by Kasner in handling his difficult mathematical material.

Next came some papers on the geodesic mapping of surfaces, wherein, incidentally, Kasner pointed out the triviality of some of the previously published work. The principal theme here is the extension in various directions of Beltrami's classic theorem that only surfaces of constant total curvature admit a mapping of their geodesics into straight lines.

A great honor that came to Kasner at this time, showing that his foremost position in American geometry was already recognized, was his being chosen as the main speaker on the subject at the International Congress of Arts and Sciences meeting at St. Louis in 1904, in connection with the World's Fair of that year. His address, "Present Problems of Geometry," was a comprehensive summary and formulation of the status of the subject at that time. Subsequent developments justified in many ways Kasner's opinion as to what was fundamental and the direction future research would probably take. The speech, published in the *Bulletin of the American Mathematical Society,* also aroused wide interest abroad; an indication of this was the publication of a translation in a Polish mathematical journal. Incidentally, one of Kasner's auditors at the St. Louis Congress was the great French mathematician Henri Poincaré, himself one of the principal invited speakers.

The main work, however, of Kasner's early period was on the Trajectories of Dynamics. This topic, in its various ramifications, was dealt with in a number of published papers, and was the subject of his Colloquium Lectures, delivered by invitation at the summer meeting of the American Mathematical Society in 1909, later published in book form by the Society.

An important part of classical dynamics deals with the orbits of particles moving in a positional field of force. Thus we have, e.g., the Newtonian field of attraction to a fixed center according to the inverse square law; the orbits are conic sections having the fixed point for focus. In the elastic field, with the attraction to a fixed center proportional to the distance, the orbits are ellipses with the fixed point as geometric center. The parabolic trajectories in a field of force constant in magnitude and direction are the most elementary example.

Kasner considered a general positional field of force, i.e., with components $\phi(x,y)$, $\psi(x,y)$ that are functions solely of position, but otherwise perfectly general. The motion of a particle of unit mass in such a field is defined by the differential equations

$$(1) \qquad \frac{d^2x}{dt^2} = \phi(x,y), \frac{d^2y}{dt^2} = \psi(x,y).$$

The point of Kasner's work was to consider the *totality* of ∞^3 trajectories arising out of all possible initial states (position and velocity) of the moving particle. This triply infinite family of curves is defined by a differential equation of third order, which one finds by elimination of the time from the equations (1) viz.:

$$(2) \qquad (\psi - y'\phi)y''' = \{\psi_x + (\psi_y - \phi_x)y' - \phi_y y'^2\}y'' - 3\phi y''^2.$$

The special form of this equation, as compared with the general differential equation of third order, $y''' = f(x,y,y'y'')$, indicates that the total family of trajectories in *any* positional field of force must have some special geometric properties. Kasner found a number of such properties, from which he selected a set of five as characteristic, i.e., necessary and sufficient conditions on a family of ∞^3 curves in the plane in order that they may be identifiable with the totality of orbits in some positional field of force.

Property I is to the effect that the ∞^1 trajectories which start from any fixed point in any fixed direction have at that point osculating parabolas whose foci form a circle passing through the fixed point. The form of differential equation expressive of this property is:

$$(3) \qquad y''' = G(x,y,y')y'' + H(x,y,y')y''^2,$$

including (2), but more general. This type of curve family was found by Kasner to occur in various other geometric problems, e.g., in the study of the plane sections of an arbitrary surface.

Kasner extended his theory to three dimensions, obtaining a characteristic set of geometric properties of the ∞^5 trajectories of an arbitrary positional field of force in space, as defined by equations

$$(4) \qquad \frac{d^2x}{dt^2} = \varphi(x,y,z), \frac{d^2y}{dt^2} = \psi(x,y,z), \frac{d^2z}{dt^2} = \theta(x,y,z).$$

Again, elimination of *t*, leaving only the space coordinates *x, y, z*, gave a system of differential equations, one of the second, one of the third order, whose special analytic form was then translated into geometric terms. For instance, the particles starting from any fixed point in any fixed direction with varying speeds describe ∞^1 curves which have the same osculating plane and the same torsion.

Generalization along various lines, to *n* dimensions, to particles constrained to move on a given surface, to the case of an additional resisting force depending on velocity as well as position, etc., were made by Kasner and his students.

Sometimes Kasner discovered, as a by-product of his work, a theorem of physical as well as mathematical interest. Thus, he showed that in any positional field of force, a particle starting from rest described a path whose initial curvature was one third that of the line of force at the starting point. This result was applied by a physicist to calculate the deflection from the vertical (eastward in the northern hemisphere) of bodies falling from a great height, due to the earth's rotation.

In a conservative field of force, the trajectories can be grouped according to the value of the energy constant *h* (kinetic + potential). This leads in the plane to a family of ∞^2 curves for each fixed value of *h*, and in space to ∞^4. The principle of least action affords a characterization of these curves as extremals of a calculus of variations problem of the form

$$(5) \qquad \int F(x,y,z)ds = \text{minimum},$$

where $F(x,y,z) = \sqrt{W(xy,z) + h}$, $W(x,y,z)$ being the work function (negative potential energy).

Another physical interpretation is to regard $F(x,y,z)$ as the index of refraction in a non-homogeneous but isotropic optical medium, whereupon the integral in (5) becomes the time of passage of light, and the minimum requirement is the statement of Fermat's principle of least time, determining the light paths. Because of this and many other physical interpretations, Kasner adopted the name "natural family" for the totality of ∞^4 extremals of a variation problem of type (5). From the Euler-Lagrange equations,

$$(6) \qquad y'' = (L_y - y'L_x)\ (1 + y'^2 + z'^2),$$
$$z'' = (L_z - z'L_x)\ (1 + y'^2 + z'^2),\ L = \log F,$$

be obtained a number of purely geometric properties of natural families, from which he selected several completely characteristic sets.

In this context belongs Kasner's work on the converse of a certain important theorem of Thomson (Lord Kelvin) and Tait. Stated in optical form (the original version was of an equivalent dynamical nature) these physicists proved that the ∞^2 paths of light which emanate orthogonally from any surface (that may reduce to a curve or a point) are always orthogonal to an entire family of ∞^1 surfaces (wave fronts), and that the time occupied in passing between any two of these surfaces is the same on all the light paths. Of course, this is now recognized to be a particular case of the Kneser transversality theory in the calculus of variations, since the transversality relation for integrals of the type (5) is orthogonality—indeed, this property is characteristic of that form of variation problem.

Kasner realized, however, that, since it is a special geometric property of a congruence of ∞^2 curves in space to admit an orthogonal family of surfaces (normal congruence), the theorem of Thomson-Tait had, beneath its physical exterior, a purely geometrical core as follows: in a natural family of ∞^4 curves, the ∞^2 curves which meet an arbitrary surface orthogonally will admit ∞^1 orthogonal surfaces, i.e., form a normal congruence. This led Kasner to formulate the converse question: If a family of ∞^4 curves in space

$$(7) \qquad y'' = F(x,y,z,y',z'),\ z'' = G(x,y,z,y',z')$$

has the property that the subfamily of ∞^2 curves orthogonal to an arbitrarily given surface forms a normal congruence, is the family necessarily of the natural type, i.e., do its differential equations specialize to the form (6)? In "The Theorem of Thomson and Tait and Natural Families of Trajectories" (1910), he proved that the answer to this converse question was in the affirmative.

His penchant for exploring all the facets of a given mathematical situation led Kasner to the formulation of many related problems by weakening the hypotheses. For instance, he raised the question: If only the congruences of ∞^2 curves, chosen from the given ∞^4, which radiate from an arbitrary point of space form a normal congruence, what can be said of the given family (7)? Kasner found by examples that the family was not necessarily natural, but the complete determination of all such curve families escaped his methods, and was cited by him in his published Colloquium Lectures and in his seminar teaching as an interesting but difficult undertaking. I still remember Kasner's warm, unstinted praise when (as a youth of twenty) I presented my own solution of this problem in Kasner's seminar and to the Columbia Mathematical Colloquium. The method used was based on equations of variation. It turned out that there were three types of curve family with the stated property: (1) natural families, (2) a type called by the writer quasi-natural, characterized by the property that every sphere gives rise to a normal congruence, (3) a third type defined by a certain system of partial differential equations involving F, G.

I mention this work as an illustration of Kasner's ability to inspire graduate students to serious mathematical research by leading them at the very beginning to interesting and difficult problems.

The sudden rise of Einstein's general theory of relativity to widespread acceptance and acclaim in 1919 followed confirmation of his prediction of light deflection in the solar gravitational field. This agreement with observation and measurement was established by the astronomical expeditions sent to Sobral in northern Brazil and to the island of Principe off the coast of Africa to observe the solar eclipse in May of that year.

With his geometrical equipment, Kasner was one of those well-prepared to follow out some of the purely mathematical aspects of the relativity theory, based, as it is, on regarding the space-time continuum as a four-dimensional Riemannian manifold. Among other things, Kasner proved that the solar gravitational field, as defined by the formulas of Schwarzschild, was completely determined by its light paths (minimal geodesics) without the need of knowing all (conceivable) planetary orbits (totality of geodesics). He also investigated Riemannian manifolds defined by separable quadratic forms, such as

$$ds^2 = Q' + Q'',$$

where, e.g., Q' involves in both coefficients and differentials only x_1, x_2 and Q'' only x_3,x_4. He determined all such forms which obeyed Einstein's equations in the cosmological form:

$$R_{ij} - \lambda g_{ij} = O,$$

where R_{ij} is the Ricci tensor, g_{ij} the metric tensor, and λ the "cosmological constant." This result was generalized to the case of "partial separability," e.g.,

$$ds^2 = Q' + Q'' + Q''',$$

where Q' involves only x_1,x_2; Q'', x_1,x_3; Q''', x_1,x_4; i.e., the separate groups of variables are permitted to overlap.

Kasner also proved that an Einstein space that was not itself Euclidean could not be immersed in a five-dimensional Euclidean space. As is well known, the Schwarzschild form, representing the solar gravitational field, can be immersed in a six-dimensional Euclidean space.

About 1927, Kasner became interested in the theory of "polygenic" functions of a complex variable. In applying the concept of derivative to any complex-valued function

$$w = P(x,y) + iQ(x,y)$$

of a complex variable $z = x + iy$, one forms the incremental ratio $\Delta w/\Delta z$ and seeks its limit as $\Delta z \rightarrow O$. One finds that the value of this limit generally depends on the direction in which the point $z + \Delta z$

tends to z; if the slope of this direction is *m,* the derivative of *w* in this direction is

$$\left(\frac{dw}{dz}\right)_m = \frac{P_x + iQ_x + (P_y + iQ_y)m}{1 + im}. \tag{8}$$

In the classic theory, one considers exclusively the case in which dw/dz is independent of the direction *m*; the condition for this is expressed by the Cauchy-Riemann equations,

$$P_x = Q_y, \qquad P_y = -Q_x, \tag{9}$$

and the functions obeying this condition are called "monogenic."

Kasner, however, proposed to consider the general case in which the value of the derivative depends effectively on the direction of approach of Δz to O; in contrast with the classic case, he used the term "polygenic" for such a function. He found that the total set of values obtained for dw/dz by varying the direction *m* had for its geometric locus a circle, which he termed the "derivative circle." The points of this circle were, moreover, in homographic correspondence with the pencil of directions *m.*

The geometric implications of these concepts were developed by Kasner and his students in a number of publications.

About 1940, Kasner returned to some of his earlier investigations on horn angles. This is the figure formed by two arcs (the sides) tangent, or having higher contact with one another, at a point (called the vertex). He found the following invariant of a horn angle under the group of conformal point transformations:

$$M_{12} = \frac{(\gamma_1 - \gamma_2)^2}{\dfrac{d\gamma_1}{ds_1} - \dfrac{d\gamma_2}{ds_2}},$$

where γ_1, γ_2 are the curvatures of the respective sides at the vertex, and d/ds_1, d/ds_2 denote differentiation as to the arc length of the respective sides.

He invented and studied the figure of the trihorn, a generalization of a plane or spherical triangle. He defined the sides and angles of such a figure, and developed a system of formulas, called by him "trihornometry," which generalized the classic formulas of plane and spherical trigonometry. Fundamental was a certain metric of the Finsler type: $ds = dx^2/dy$.

During the last twenty years of his research career, Kasner collaborated extensively with his student John De Cicco. The two formed an ideal research combination—Kasner with his lively imagination and instinct for fruitful problems, and De Cicco with his great power and technical proficiency in dealing with geometric problems. On summer vacations from his teaching duties at the Illinois Institute of Technology and later at the De Paul University in Chicago, De Cicco would often visit Kasner for several weeks, and their scientific conversations were always productive of a number of joint papers of high merit.

The bibliography at the end of this memoir was compiled by Professor De Cicco in characteristic thorough fashion, and I take this occasion to express acknowledgment and thanks. It is a complete or nearly complete list of Kasner's mathematical publications over the more than half-century span, 1900–1952, of his research activity.

When the subject of this biographical sketch began his mathematical career at the turn of the century, America occupied a minor position in world mathematics. If today this position is patently a leading one, then some significant portion of the credit must be assigned to the work and influence of Edward Kasner.

KEY TO ABBREVIATIONS

Amer. Jour. Math. = American Journal of Mathematics
Amer. Math. Jour. = American Mathematical Journal
Amer. Math. Mo. = American Mathematical Monthly
Ann. Math. = Annals of Mathematics
Bull. Amer. Math. Soc. = Bulletin of the American Mathematical Society
Duke Math. Jour. = Duke Mathematical Journal
Jour. Math. Phys. = Journal of Mathematics and Physics
Jour. Philos. = Journal of Philosophy
Math. Ann. = Mathematische Annalen
Nat. Math. Mag. = National Mathematics Magazine
Proc. Fifth Internat. Cong. Math. = Proceedings of the Fifth International Congress of Mathematics, Cambridge
Proc. Nat. Acad. Sci. = Proceedings of the National Academy of Sciences
Sci. Mo. = Scientific Monthly
Trans. Amer. Math. Soc. = Transactions of the American Mathematical Society

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