Oscar Zariski

1899–1986

BIOGRAPHICAL *Memoirs*

A Biographical Memoir by
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NATIONAL ACADEMY OF SCIENCES

OSCAR **ZARISKI**

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Elected to the NAS, 1944

By David Mumford

Oscar Zariski was born in the town of Kobryn, which lies on the border between Poland and Russia. It was a part of Russia at the time of Zariski's birth, a part of Poland between the two world wars, and is now in the Republic of Belarus. Zariski was the son of Bezalel and Chana Zaristky, and was given the name of Asher Zaristky, which he changed to Oscar Zariski when he later came to Italy. Kobryn was a small town where his mother ran a general store, his father having died when he was two. In 1918, Zariski went to the University of Kiev in the midst of the revolutionary struggle. He was seriously wounded in one leg[1] when caught in a crowd that was fired upon by troops, but he recovered after two months in the hospital. As a student, he was attracted to algebra and number theory as well as to the revolutionary political ideas of the day. Zariski supported himself partly by writing for a local Communist paper. This is most surprising for those of us who only knew him much later, but it calls to mind the quip, "A man has no heart if he is not radical in his youth and no mind if he is not conservative in his mature years."

In 1921, Zariski went to Italy, largely because of the limited educational opportunities available to Jews in Russia at that time. First he enrolled at the University of Pisa but after six months switched to the University of Rome, where the famous Italian school of algebraic geometry—led by Guido Castelnuovo, Federigo Enriques, and Francesco Severi—was flourishing. The fact that universities in Italy were free to foreign students was an important consideration, as Zariski had no money. He was especially attracted to Castelnuovo, who recognized Zariski's talent immediately. As Zariski told it, Castelnuovo took him on a three-hour walk around Rome, after which Zariski realized that he had been given an oral exam in virtually every area of mathematics. He also worked closely with the more colorful Enriques who, however, had a cavalier attitude toward contem-

1 The bullet migrated to the surface 60 years later, while he was visiting the University of California, Berkeley.

porary standards of mathematical argument; he said, “We aristocrats do not need proofs; proofs are for you plebians.” Castelnuovo saw in Zariski a man who could not only bring more rigor to their subject and push it deeper but also could find fundamentally new approaches to overcoming its limitations. Zariski was fond of quoting Castelnuovo as saying, “Oscar, you are here with us but are not one of us,” referring to Zariski’s interest in algebra and to his doubts even then of the rigor of their proofs.

The student Zariski met his wife, Yole Cagli, while in Rome and they were exchanging lessons in Italian and Russian. They were married in 1924, in Kobryn, and he received his doctorate that same year.

His thesis (Zariski 1926) classified the rational functions $y=P(x)/Q(x)$ of x such that 1) x can be solved for in terms of radicals starting with y, and 2) given distinct solutions x_1 and x_2, all other solutions x are rational functions of x_1 and x_2. Already in this first work he showed his ability to combine algebraic ideas (the Galois group), topological ideas (the fundamental group), and the “synthetic” ideas of classical geometry. The interplay of these different tools characterized his life’s work.

Zariski pursued his approaches with the support of a Rockefeller fellowship in Rome during the years 1925 to 1927. His son Raphael was born there in 1925. Because Fascism under Mussolini was growing more invasive in Italy at that time, Zariski felt it would be wise to move again. Castelnuovo put him in touch with Solomon Lefschetz, who had moved to the United States and was teaching at Princeton University. Lefschetz helped Zariski find a position at Johns Hopkins, which he accepted in 1927. In 1928 his family joined him in Baltimore, where in 1932 his daughter Vera was born.

An important paper early in Zariski’s career was his analysis (Zariski 1928) of an incomplete proof by Severi that the Jacobian of a generic curve of genus 9 has no nontrivial endomorphisms. Severi’s paper read as though the proof were complete, but Zariski discovered a problem and found an ingenious argument to remedy it. Neither were well received by Severi, who published his own correction independently.

This discovery, plus his learning of Lefschetz’s groundbreaking work in topology, influenced Zariski deeply, and he began to study the topology of algebraic varieties, especially their fundamental groups (Zariski 1931, 1932). The techniques applied here were rigorous beyond a doubt, and the tools were clean and new. He traveled frequently to Princeton to discuss his ideas with Lefschetz. In this phase of his career, roughly from 1927 to 1935, Zariski studied the fundamental group of a variety through the fundamental group of the complement of a divisor in projective *n*-space. This work was characterized by its spirit of exploration and discovery and, in spite of much recent interest, it remains largely uncharted. One result gives the flavor of the new things he discovered: All plane curves of fixed degree with a fixed number of nodes belong to a single algebraic family. (This was the result of another incomplete paper of Severi, and a correct proof was found only many years later.) What Zariski found was that curves with a fixed degree and a fixed number of cusps, the next most complicated type of double point, could belong to several families. He exhibited curves C_1 and C_2 of degree 6 with 6 cusps such that the fundamental groups of their complements were not isomorphic.

He saw clearly that the lack of rigor he had previously touched on wasn't limited to a few isolated sores but was actually a widespread disease. His goal now became the restoration of the main body of algebraic geometry to proper health.

In 1935, Zariski completed the monograph *Algebraic Surfaces*, his monumental review of the central results of the Italian school (Zariski 1935). His goal had been to disseminate more widely the ideas and results of his teachers, but one result for him was "the loss of the geometric paradise in which I so happily had been living."[2] He saw clearly that the lack of rigor he had previously touched on wasn't limited to a few isolated sores but was actually a widespread disease. His goal now became the restoration of the main body of algebraic geometry to proper health. Algebra had been his early love and algebra was blooming, full of beautiful new ideas in the hands of Wolfgang Krull and Emmy Noether; and various applications to algebraic geometry had already been proposed by B. L. van der Waerden. Zariski said later, "It was a pity that my Italian teachers never told me that there was such a tremendous development of the algebra which is connected with algebraic geometry. I only discovered this much later when I came to the United States."[3] Zariski threw himself into this newly enriched discipline. He spent the year 1935–1936 at the Institute for Advanced Study in Princeton, and met regularly with Noether, then at Bryn Mawr College, learning the new field through firsthand contact with the master.

2 Zariski, O. 1972. *Collected works of Oscar Zariski*, Vol. I, edited by D. Mumford and H. Hironaka. Cambridge, MA: MIT Press, Preface.

3 Parikh, C. 2009. *The Unreal Life of Oscar Zariski*. New York: Springer Science+Business Media.

The 15 years or so that followed, 1938 to 1951—if you take the years between his paper recasting the theory of plane curve singularities in terms of valuation theory (Zariski 1938) and his awe-inspiring treatise (Zariski 1951) on "holomorphic functions" (not the usual use of the term but here referring to the limits of sequences of functions that converge in an I-adic topology)—saw an incredible outpouring by Zariski of original and creative ideas; he took tool after tool from the kit of algebra and applied them to elucidate basic geometric ideas. Though many mathematicians in their 40s reap the benefits of their earlier more-original work, Zariski undoubtedly was at his most daring exactly in this decade. He corresponded extensively at this time with André Weil, who was also interested in rebuilding algebraic geometry and extending it to characteristic *P*, with a view to its number-theoretic applications in this case. One of the main themes of this period for both of them was extending algebraic geometry to work over an arbitrary ground field. At that time, Zariski called this greater theory "abstract" algebraic geometry, though such an adjective now seems dated. Although they rarely agreed, they found each other very stimulating, Weil saying later that Zariski was the only algebraic geometer whose work he trusted. They managed to be together in 1945 while both were visiting the University of São Paulo in Brazil.

At the same time, these were years of terrible personal tragedy. During the war, all of Zariski's relatives in Poland were killed by the Nazis. Only his own family and the families of two siblings who had moved to Israel escaped the Holocaust. He told the story of how he and Yole were halfway across the United States, driving to the East Coast, the day Poland was invaded. They listened each hour to the news broadcasts on their car radio, their only link to the nightmare half a world away. There was nothing they could do.

In this period, Zariski solved many problems with algebraic methods that were new at the time, and which he made his own. He gave a survey talk (Zariski 1950a) on the new perspective to the first postwar International Congress of Mathematics, held in Cambridge, Massachusetts. Three themes in his work were particularly beautiful and deep, so we want to describe them here in some detail.

The first theme was his study of birational maps, which led him to the famous result universally known as "Zariski's Main Theorem." This was the final result in a foundational analysis of birational maps between varieties (Zariski 1943). Birational maps are algebraic correspondences that are bijective at most places but may "blow up" or "blow down" at special points. Zariski showed that if there are points P and Q in the range and domain that are isolated corresponding points—i.e., the set of points corresponding to *P* contains *Q* but no curve through *Q*, and vice versa—and if, further, *P* and *Q* satisfy an algebraic restriction (that is, they are normal points)—then in fact *Q* is the only point corresponding to *P* and the map is biregular between *P* and *Q*. Zariski's proof of this hypothesis was astonishingly subtle, yet short.

The concept of integral extensions and normality had proved essential in algebraic number theory, and it had been extended to noetherian rings in the 1930s. In Zariski's hands, it became a major tool in algebraic geometry. The "Main Theorem" asserts in a strong sense that the normalization (the integral closure) of a variety X is the maximal variety X' birational over X, such that the fibres of the map $X' \to X$ are finite. A generalization of this fact became Alexandre Grothendieck's concept of the "Stein factorization" of a map.

Zariski's second major theme of this period was the resolution of singularities of algebraic varieties (Zariski 1939, 1940, 1944), which culminated in his proof that all varieties of dimension at most 3 (in characteristic zero) have "nonsingular models"—i.e., these varieties are birational to nonsingular projective varieties. This was a problem that had totally eluded the easygoing Italian approach. Even in the case of dimension 2, although some classical proofs were essentially correct, many of the published treatments were not. Zariski attacked this problem with a whole battery of techniques, pursuing it relentlessly over six papers and 200 pages. Perhaps his most striking new tool was the application of the theory of general valuations in function fields to give a birationally invariant way to describe the places that must be desingularized. Here again, though the structure of valuations had been investigated before, they came to life in Zariski's hands and took their rightful place in algebraic geometry. Though to some extent shunted to the sidelines in succeeding years, no doubt they will rise again.

Zariski's results proved to the mathematical world the power of the new ideas. For many years, this work was also considered to be technically the most difficult in all of algebraic geometry. Only when the result was proven for surfaces in characteristic *p* by S. S. Abhyankar and later for varieties of arbitrary dimension in characteristic 0 by Heisuke Hironaka[4] was this benchmark surpassed.

4 Abhyankar, S. S. 1956. Local uniformization on algebraic surfaces over ground fields of characteristic $p \neq 0$. *Annals of Math.* 63(3): 491–526; Hironaka, H. 1964. Resolution of singularities of an algebraic variety over a field of characteristic 0. *Annals of Math*. 79(1):109–203.

Zariski's third theme was his theory of abstract "holomorphic functions" (Zariski 1948, 1951). The idea here was to use the notion of formal completion of rings with respect to powers of an ideal as a substitute for convergent power series, and to put elements of the resulting complete rings to some of the same uses as classical holomorphic functions. Because the theory of completions was less well developed at the time, Zariski wrote several foundational papers on the subject in preparation for this work, including the development of a theory of I-adic completions of noetherian rings with respect to arbitrary ideals I. His work on holomorphic functions was immediately recognized as fundamental, though its most striking application at the time was to a stronger version of the "Main Theorem," known as the "Connectedness Theorem," which states that the fibres of a birational morphism from a projective variety X to a normal variety Y are connected. Later, in the hands of Grothendieck, Zariski's theory became a central tool of algebraic geometry.[5]

Whenever Harvard's baroque appointment rules, known as the [William] Graustein Plan after the earlier mathematician who invented them, jibed with his own plans, he used them; but whenever they did not, he feigned ignorance of all that nonsense and insisted the case be considered on its own merits.

In 1945, Zariski moved to a research professorship at the University of Illinois. But early in the '40s, his work had caught the attention of G. D. Birkhoff, who decided Zariski must come to Harvard University,[6] and indeed in 1947 he did so and remained there for the rest of his life. He had been the first Jew to join Harvard's math department, but he fit in well and enjoyed its still-formal ways.[7] Moreover, he exerted a very strong influence on the mathematical environment there and enjoyed luring the best people he could to the department and bringing out the best in his students. While Zariski was chairman, Dean McGeorge Bundy used to refer to him as that "Italian pirate," so clever was he in getting his way, inside or outside the usual channels. Whenever Harvard's baroque

5 Grothendieck's style was the opposite of Zariski's. Whereas Zariski's proofs always had a punch line, a subtle twist in the middle, Grothendieck would not rest until every step looked trivial. In the case of holomorphic functions, Grothendieck liked to claim that the result was so deep for Zariski because he was proving it for the 0th cohomology group. The easy way, he said, was to prove it first for the top cohomology group, then use descending induction!

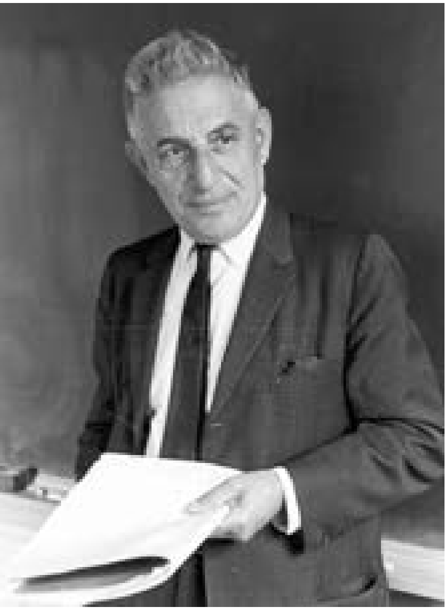
6 The story, which we have heard from reliable sources, is that Birkhoff approached Zariski and said in his magisterial way, "Oscar, you will probably be at Harvard within the next five years."

7 Even 15 years later, at dinner parties the women would leave the table after dessert while the men smoked cigars.

appointment rules, known as the [William] Graustein Plan after the earlier mathematician who invented them, jibed with his own plans, he used them; but whenever they did not, he feigned ignorance of all that nonsense and insisted the case be considered on its own merits.

Over some 30 years, Zariski made Harvard into the world center of algebraic geometry. His seminar welcomed Wei-Liang Chow, Grothendieck, W. V. D. Hodge, Jun-Ichi Igusa, Kunihiko Kodaira, Masayoshi Nagata, Jean-Pierre Serre, Weil, and many others. The stimulating evenings at Oscar and Yole's home and the warm welcomes they extended are not easily forgotten. Zariski's reconstruction of algebraic geometry began with his writing of the monograph *Algebraic Surfaces*, and once Zariski felt he had reliable and powerful general tools, it was natural for him to see if he could put all the main results of the theory of surfaces in order. Here is a (necessarily incomplete) list of some topics he reworked: (i) the relationship of geometric nonsingularity with the algebraic concept of regularity (Zariski 1947); (ii) the basics of linear systems (Zariski 1950b, 1962); (iii) vanishing theorems for cohomology (specifically what he called the "lemma of Enriques-Severi," before the topic was taken up by Serre and Grothendieck) (Zariski 1952); (iv) the questions of the existence of minimal nonsingular models in each birational equivalence class of algebraic surfaces (Zariski 1958a); (v) the classification of varieties following Castelnuovo and Enriques (Zariski 1958b) (now known as the classification by Kodaira dimension); and (vi) the codimension of the branch locus (Zariski 1958c). In each of these areas he spread before his colleagues and students the vision of many areas to explore and a broad array of exciting prospects. He wrote a now -classic textbook, *Commutative Algebra*, aimed at the geometric applications that he had pioneered (Zariski 1958d, 1960).

In the course of this work, Zariski developed a fully worked-out approach to the foundations of algebraic geometry and indeed had written it up in a book manuscript as a sequel to *Commutative Algebra*. But in welcoming the appearance of yet newer definitions and techniques that were making the subject stronger, he never published his own version of the foundations. Thus he embraced the new language of sheaf theory and cohomology, and he worked through the basic ideas methodically in the Summer Institute in Colorado in 1953 (Zariski 1956) as was his custom, although he never adopted this language as his own. When Grothendieck appeared, Zariski immediately invited him to Harvard. Grothendieck, for his part, welcomed the prospect of working with Zariski. Grothendieck's political beliefs did not allow him to swear the oaths of loyalty required in those politicized days; he asked Zariski to investigate the feasibility



Oscar Zariski, approximately 1979.

of continuing his mathematical research from a Cambridge jail cell—e.g., how many books and visitors would he be allowed!

The final phase of Zariski's mathematical career was a return to the study of singularities (Zariski 1965, 1966, 1968, 1971, 1975). He had absolutely no use for the concept of retirement, and he dedicated his 60s and 70s and as much of his 80s as he could to a broad-based attack on the problem of "equisingularity." The goal was to find a natural decomposition of a variety X into pieces Y, each one made up of a subvariety of X from which a finite set of lower-dimensional subvarieties have been removed, such that along each subvariety Y_i the variety X had essentially the same type of singularity at every point. Zariski made major strides toward the achievement of this goal, but the problem is difficult and work is still in progress.

Honors flowed to Zariski in well-deserved appreciation of the extraordinary contributions he had made to the field of algebraic geometry. He was elected to the national Academy of Sciences in 1944. He received the Cole Prize from the American Mathematical Society in 1944, the National Medal of Science in 1965, and the Wolf Prize in 1982; and he was awarded honorary degrees from the College of the Holy Cross in 1959, Brandeis University in 1965, Purdue University in 1974, and Harvard in 1981.

Zariski's last years were disturbed by his fight with hearing problems. He had always been very lively and astute, both in mathematical deliberations and in social interactions, picking up every nuance; but late in life he was struck with tinnitus (a steady ringing in the ears), as well as a greater sensitivity to noise and a gradual loss of hearing. These disorders forced Zariski into himself and into his research, keeping him close to home. Only the boundless devotion of his family sustained him in those last years. He died at home on July 4, 1986. Zariski's colleagues, students, and friends will remember not only the beautiful theorems he found but also the forcefulness and warmth of the man they knew and loved.

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