# Jesse Douglas

1897–1965

BIOGRAPHICAL *Memoirs*

*A Biographical Memoir by*
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NATIONAL ACADEMY OF SCIENCES

# JESSE **DOUGLAS**

*July 3, 1897–September 7, 1965*

Elected to the NAS, 1946

Jesse Douglas was a mathematician best known for solving the *Plateau problem* in geometry. He was one of the first two recipients of the newly established Fields Medals in 1936. The *Plateau problem*, first raised by Joseph-Louis Lagrange in 1760, asks whether every closed curve spans a surface of least area. This is the mathematical model for soap films, where nature tries to minimize the area of a soap film spanning a wire contour. The *Plateau problem* was extensively studied by the experimental physicist Joseph Plateau in the 19th century. Douglas proved mathematically that a minimizer exists.

*By Tobias Holck Colding and William P. Minicozzi*

## Early Life and Education

Jesse Douglas was born in New York on July 3, 1897, to Louis and Sarah Kommel Douglas and developed an interest in mathematics in high school. He studied at the City College of New York, winning the Belden Medal for excellence in mathematics in his first year and graduating with honors in mathematics in 1916. He went on to graduate school at Columbia University under the supervision of Edward Kasner, submitting a doctoral thesis entitled “On Certain Two-Point Properties of General Families of Curves; The Geometry of Variations” in 1920. Douglas continued to study differential geometry while teaching at Columbia College from 1920 to 1926.*

In 1926, he earned a four-year National Research Fellowship and spent time at Princeton University (1926–27), Harvard University (1927), Chicago (1928), Paris (1928–30), and Göttingen (1930). Throughout this period, he worked on the *Plateau problem*, which had

* Gray, J., and M. Micallef. 2008. About the cover: The work of Jesse Douglas on minimal surfaces. *Bull. Amer. Math. Soc.* 45(2):293–302.

Rassias, T. M., ed. 1992. *The Problem of Plateau. A Tribute to Jesse Douglas and Tibor Radó*. Singapore: World Scientific.

been posed by Lagrange in 1760 and then had been studied by leading mathematicians such as Berhard Riemann, Karl Weierstrass, Hermann Schwarz, and Henri Lebesgue as well as the Belgian physicist for which it is named, Joseph Plateau. In a series of papers from 1927 to 1931, Douglas detailed the complete solution, culminating in his 1931 paper.[1] In recognition of this work, he was awarded one of the two first Fields Medals at the International Congress of Mathematicians at Oslo, Norway, in 1936.

In 1930, Douglas was appointed an assistant professor of mathematics at the Massachusetts Institute of Technology (MIT). He was promoted to associate professor in 1934 and spent the year 1934–35 as a research fellow at the Institute for Advanced Study at Princeton University. He was a member of the faculty at the Institute for Advanced Study from 1938 to 1939. He received Guggenheim Foundation Fellowships in 1940 and 1941, then taught at Brooklyn College and Columbia University. Douglas married Jessie Nayler on June 30, 1940; they had one son, Lewis Philip Douglas.

The solution of the *Plateau problem* led to many other important questions, and Douglas continued to do foundational work in this direction. He wrote a series of important papers in the 1930s on minimal surfaces and, in 1943, he was awarded the Bôcher Memorial Prize by the American Mathematical Society for this work. The award specifically recognized three papers from 1939.[2,3,4] He was elected to the National Academy of Sciences in 1946. In 1955, Douglas was appointed professor of mathematics at the City College of New York, where he remained until his death in 1965.

## The *Plateau Problem*

Jesse Douglas came to prominence by solving a fundamental problem in geometry that had long eluded experts. This problem was first raised by Joseph-Louis Lagrange in 1760 and was later named the “*Plateau problem*” by Lebesgue:

Given a closed curve $\gamma \subset \mathbf{R}^3$, find a surface $\Sigma$ of least area with boundary $\gamma$.

Belgian physicist Plateau studied this problem experimentally, using soap films, in the mid-nineteenth century. Plateau dipped a curved wire into a soapy solution and noted that when he removed the wire, a film spanned the wire curve. Surface tension pulled the film tight, decreasing its area as much possible. This area-minimization problem was known as the *Plateau problem*.

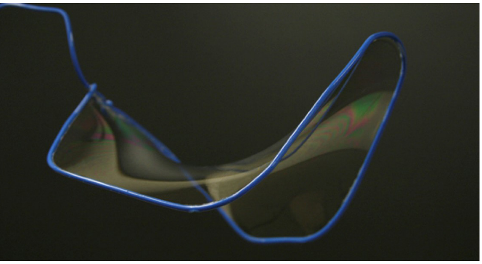

Figure 2. Surface tension pulls the soap tight, minimizing its area.

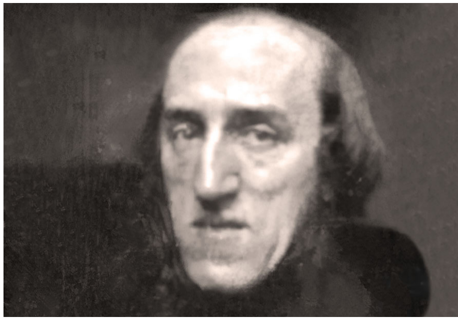

Figure 3. Joseph Plateau. (Photograph by Albert Callisto, CC BY-SA 4.0, via Wikimedia Commons)

The least area surface is called a *minimal surface*. The *Plateau problem* can be viewed as the two-dimensional generalization of the geodesic problem of finding the shortest path connecting two fixed points.

**Minimal surfaces.** The study of minimal surfaces goes back to Leonhard Euler and Lagrange and the very beginning of the calculus of variations. The starting point is to define a local quantity, called the mean curvature, that measures the average bending of a surface $\Sigma$. To do this, choose a function $u$ on $\mathbf{R}^3$ so that $u = 0$ on $\Sigma$ and $\nabla u \neq 0$ on $\Sigma$. Since the gradient $\nabla u$ is always orthogonal to level sets $\{u = \text{constant}\}$, so it follows that $\frac{\nabla u}{|\nabla u|}$ is a unit normal to $\Sigma$. The mean curvature of $\Sigma$ is then given by

$$H = \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) = \frac{\Delta u}{|\nabla u|} - \frac{\langle \nabla |\nabla u|, \nabla u \rangle}{|\nabla u|^2}\,. \tag{2.1}$$

For example, when $\Sigma$ is a unit sphere, we can choose

$$u = x^2 + y^2 + z^2 - 1$$

and compute that $H = 2$. The mean curvature measures the average bending of a surface over all possible directions. A flat plane has zero mean curvature, but many curved surfaces also have zero mean curvature.

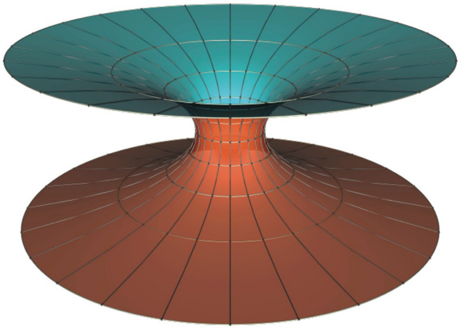

Figure 4. The catenoid is a minimal surface that has a rotational symmetry. It was discovered by Euler in 1741.

Lagrange argued that if an area-minimizing surface exists, then its mean curvature $H$ will vanish at each point. This was the beginning of the calculus of variations, and the idea is a classical calculus of variations argument. We place the minimizer $\Sigma$ as point $\Sigma_0$ in a one-parameter family $\Sigma_s$ of surfaces with the same fixed boundary. It follows that the function of one variable given by mapping s to the area of $\Sigma_s$ must have a minimum at $s = 0$. If we vary perpendicularly to the surface with speed $u$ (which can vary with the point), then a calculation, known as the "first variation formula"[5] shows that

$$\frac{d}{ds}\Big|_{s=0} \operatorname{Area}(\Sigma_s) = \int_{\Sigma_0} u\, H\,. \tag{2.2}$$

In particular, because Area($\Sigma_s$) is minimized at $s = 0$, the first derivative test says that

$$\int_{\Sigma_0} u\, H = 0\,. \tag{2.3}$$

If (2.3) holds true for any such variation, that is, for any u that vanishes at the boundary, then an easy argument shows that $H$ must vanish as well. In fact, we see that $H$ vanishes any time that $\Sigma$ is a critical point for area. It was not necessary for it to be a minimum. A surface with zero mean curvature is defined as a minimal surface. Because a critical point is not necessarily a minimum, the term minimal is misleading although time-honored.

Physically, the mean curvature is the mathematical formulation of the force from surface tension. Thus, if surface tension is the only force acting on a membrane, then it is in equilibrium precisely when it is a minimal surface.

The two fundamental questions in minimal surface theory are:

- Existence: Can we produce minimal surfaces with various properties?
- Structure: If we have a minimal surface, what are its properties?

The structure theory for minimal surfaces is an enormous subject that remains very active today but will largely be omitted here.

The early results in existence theory came from using an ansatz to find explicit minimal surfaces with additional properties. For example, in 1741 Euler constructed a rotationally symmetric minimal surface called the catenoid, and in 1776 Jean Baptiste Meusnier found a periodic minimal surface called the helicoid. In the mid-nineteenth century, Riemann found a more complicated family of periodic minimal surfaces now known as the Riemann examples. Each of these examples had very special properties, but they turned out be surprisingly fundamental.

In the early stages, subtle issues about the spaces of surfaces and more general definitions of area made the existence problem unapproachable in general. But, starting in the mid-nineteenth century, the *Plateau problem* generated a great deal of interest among mathematicians. Early results were produced for various special cases, such as polygonal curves $\gamma$, due to the efforts of Riemann, Schwarz, and Weierstrass. One of the difficulties is that the problem was not really stated precisely from a modern perspective. To start with, which curves are allowed? Do they have to be smooth? Do they have to be rectifiable (essentially, of finite length)? And which surfaces are allowed? At the time, the surfaces were given by parameterized maps from a Euclidean disk, but how smooth do these maps have to be? What is the notion of area when the map isn't smooth? The question of a general notion of area led Lebesgue to construct the "Lebesgue integral" in his 1902 dissertation "Intégrale, Longueur, Aire." The last chapter of this thesis was on the *Plateau problem*.

**Connection with the Dirichlet problem.**

The *Plateau problem* is closely connected to the Dirichlet problem for harmonic functions, named for Peter Gustav Dirichlet. A function $w(u, v)$ is harmonic if it satisfies the Laplace equation

$$\frac{\partial^2 w}{\partial u^2} + \frac{\partial^2 w}{\partial v^2} = 0\,. \tag{2.4}$$

This differential equation arises as the Euler-Lagrange equation for the energy of a function w on a domain Ω given by

$$\int_\Omega |\nabla w|^2 = \int_\Omega \left\{ \left(\frac{\partial w}{\partial u}\right)^2 + \left(\frac{\partial w}{\partial v}\right)^2 \right\} du\, dv\,, \tag{2.5}$$

where $\nabla w$ is the gradient of $w$. In particular, if a smooth function w minimizes energy among all functions with the same boundary values, then it must satisfy the Laplace equation and, thus, is harmonic. In fact, the converse is also true: A smooth harmonic function automatically minimizes energy.

The Dirichlet problem is to find a harmonic function with given boundary values. The Dirichlet problem plays an important role in complex analysis because the energy functional is conformally invariant in dimension two. The link to the *Plateau problem* is that each of the coordinate functions for a solution of the *Plateau problem* will simultaneously give harmonic functions.

Around 1900, Lebesgue stated the *Plateau problem* in a mathematically precise way:[6]

**Problem 2.6** (Plateau Problem). Given a Jordan curve $\gamma \subset \mathbf{R}^3$, determine three functions $x(u,v), y(u,v), z(u,v)$ with the following properties

(1) $x, y, z$ are each harmonic for $u^2 + v^2 < 1$.
(2) The map $(x, y, z)$ is *almost conformal*, i.e, $x_u x_v + y_u v_v + z_u z_v = 0$ and $x_u^2 + y_u^2 + z_u^2 = x_v^2 + y_v^2 + z_v^2$ at every point in $u^2 + v^2 < 1$.
(3) $x, y, z$ are continuous on the closed disk $\{u^2+v^2 \le 1\}$ and map the circle $\{u^2+v^2 = 1\}$ injectively onto $\gamma$.
(4) The area of the image of $(x, y, z)$ is the minimum possible.

The most obvious approach to the *Plateau problem* would be to take a sequence of surfaces whose areas approach the minimal value and then try to get a subsequence of these to converge to some limiting surface that achieves the minimum. This scheme is known as the direct method in the calculus of variations. The direct method fails for the *Plateau problem*, however, because the total area of a surface gives very little a priori control over the surface itself. Because area is a geometric quantity that does not depend on how the surface is parameterized, an area bound gives no control over the parameterization of the surface.

**Solution of the *Plateau problem.*** The *Plateau problem* was finally solved by Douglas and Tibor Radó, independently, around 1930.[7] They proved that a solution to 2.6 exists for any contour $\gamma$ that bounds some continuous surface with finite area. The two mathematicians took very different approaches to the problem, with Radó's work using more traditional conformal methods, and Douglas introducing a new functional A(g) that optimized the parameterization of the boundary curve $\gamma$.

Given boundary functions $g = (g_1, g_2, g_3)$ of $\theta$ on the unit circle, the Douglas functional is

$$A(g) = \frac{1}{4\pi} \int_0^{2\pi} \int_0^{2\pi} \left( \frac{\sum_{i=1}^{3} |g_i(\theta) - g_i(\phi)|^2}{4 \sin^2 \frac{\theta - \phi}{2}} \right) d\theta \, d\phi \, . \tag{2.7}$$

This is a singular integral that depends on the values of the map on the boundary but not on their derivatives. This made the existence of a minimizer more tractable, even with the methods available at that time. The existence still required understanding an important subtlety coming from the underlying conformal invariance.[8]

The expression for the functional A($g$) in terms of the boundary values is complicated, but it takes a very simple form in terms of the harmonic extensions of the boundary values. It is just one-half of the energy of the harmonic extensions. Remarkably, Douglas showed that the harmonic extensions (i.e., the solutions of the corresponding Dirichlet problems) for the boundary values given by a minimizer of (2.7) are then automatically almost conformal and, moreover, minimize the area.

One of the interesting aspects of Douglas' approach is that it works for curves in Euclidean spaces of any dimension. For example, it produces a least-area mapping from the two-dimensional disk to $R^n$ for any $n$. Furthermore, Douglas observed that the case n = 2 of his theorem actually gives a solution to the Riemann conformal mapping problem, thereby achieving new proofs for the classical theorems of Riemann and Osgood-Carathéodory.[9]

Douglas continued to make major contributions in this area over the next decade. Three of these papers from 1939 were recognized when he was awarded the Bôcher Memorial Prize by the American Mathematical Society in 1943. In his 1939 article, Douglas proved the existence of a minimal surface bounding a collection of k non-intersecting Jordan curves in $R^n$ with prescribed genus and orientation.[10] He extended this in his follow-up article to allow for infinite genus and finite and infinite Jordan curves.[11]

In 1943, he was awarded The Bôcher Prize by the American Mathematical Society, which recognized his work on the inverse problem.[12] In particular the award was for three papers all published in 1939: Green's function and the problem of Plateau and The most general form of the problem of Plateau published in the *American Journal of Mathematics* and Solution of the inverse problem of the calculus of variations published in the *Proceedings of the National Academy of Sciences U.S.A.*[10,11,12]

In 1951, Douglas made important contributions to the theory of groups in a series of papers published in the *Proceedings of the National Academy of Sciences U.S.A.* In particular, he contributed to the study of groups with two generators a and b so that each element can be expressed at $a^r b^s$ for integers $r$ and $s$.

## Douglas' Legacy

The solution of the *Plateau problem* was a monumental achievement, settling an old problem and leading to many natural questions. Douglas himself made a number of important contributions in this direction, including solving a generalized *Plateau problem* for different topological types of surfaces, and his ideas played a role in many of the future developments. Generalizations of the *Plateau problem* continued to play an important role in mathematics for decades.

In the late 1930s, Marston Morse and Charles Brown or C. B. Tompkins, and Max Shiffman independently, used Morse theory to find higher index minimal surfaces. They proved the existence of minimal surfaces that are not even locally minimizing, but rather are critical points for area that are unstable; that is, there are ways to deform the surface to have less area. Morse and Tompkins relied on the existence of minimizers proven by Douglas and Radó for their work. In 1948, Charles Morrey solved the *Plateau problem* for curves in a Riemannian manifold, showing the existence of least-area surfaces in a very general class of curved spaces.[13]

The Douglas solution of the *Plateau problem* gives a parameterized disk where each of the components of the map is a harmonic function. It is then extremely natural to ask how regular the map is. Each component function is completely smooth (i.e., infinitely differentiable) because of the regularity theory for harmonic functions (the Weyl lemma, in this case). Moreover, the components together give a map that is almost conformal, which leaves open the possibility that the map has singularities at points where the differential vanishes ("branch points"). It might also have self-intersections where it fails to be embedded. Understanding these two issues took five more decades.

It turned out that interior branch points do not occur for the Douglas solution. A major step on this came in Robert Osserman's 1970 paper showing that so-called "true banch points" cannot occur for a minimizer, and "false branch points" were ruled out shortly afterwards by Hans W. Alt and Robert Gulliver. The question about boundary branch points remains open almost 100 years after Douglas' work, although some important cases exist.

There are some boundary curves γ for which the Douglas solution simply cannot be embedded. For example, if γ is knotted, like the trefoil curve, then there are no embedded topological disks with this boundary, and, thus, the Douglas solution must cross itself in this case. The guiding principle, however, is that the Douglas solution should be "as embedded as possible." Around 1980, William Meeks and Shing-Tung Yau proved that if γ lies on the boundary of a convex domain, then the Douglas solution must be embedded. Their ideas carried over to Riemannian three-manifolds and played a role in three-manifold topology.

The connection between area and energy breaks down after dimension two, so an entirely new approach had to be developed to find higher-dimensional area-minimizers. Three different approaches were developed around 1960 by Ennio de Giorgi, Herbert Federer and Wendell Fleming, and Ernst Reifenberg. In 1960, Federer and Fleming developed the theory of normal and integral currents and proved the existence of k-dimensional rectifiable area-minimizers in $R^n$.[14] The corresponding regularity theory was a major problem in the last century that led to the development of powerful tools now used in many areas of mathematics. For hypersurfaces, the area-minimizers turned out to be regular up until dimension seven and, in general, have a singular set that is at least seven dimensions smaller than the hypersurface. The boundary regularity was established by Robert Hardt and Leon Simon in 1979, giving an analog of Douglas' theorem for hyper-surfaces in all dimensions, but with singularities in high dimensions. The situation is more complicated in higher codimension, where Frederick J. Almgren showed that the singular set had codimension at least two; this is easily seen to be sharp for complex algebraic varieties.

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