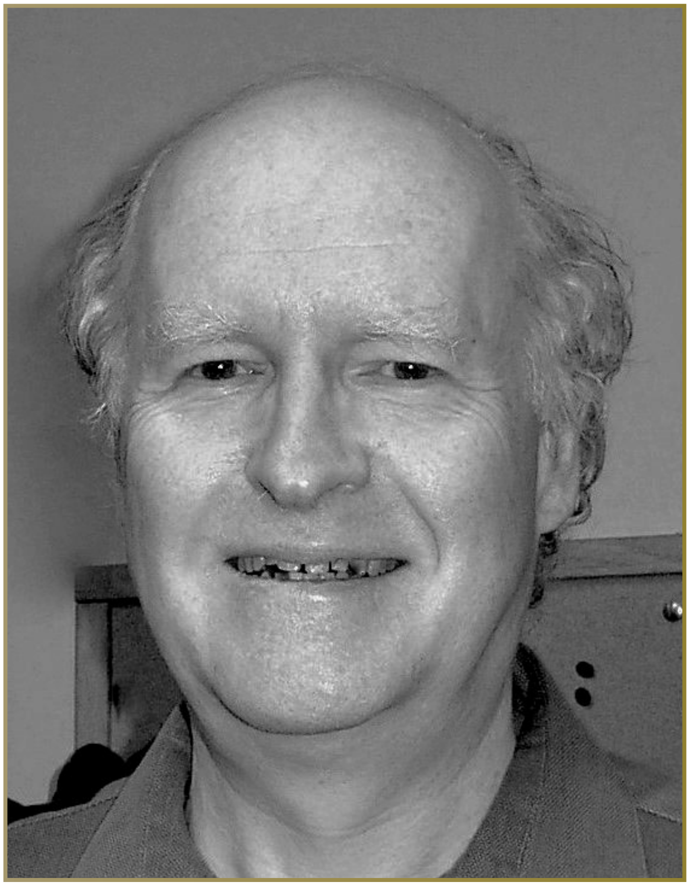
John N. Mather

June 9, 1942–January 28, 2017
Elected to the NAS, 1988

A Biographical Memoir by Mark Goresky



John Mather. *With permission: Princeton University.*

John Norman Mather was a renowned mathematician who made important contributions to the fields of singularity theory and mechanics. The theorems he developed for understanding singularities of smooth mappings, for modeling singular spaces and for explaining the chaotic behavior of orbits of frictionless mechanical systems were among the most important contributions to mathematics in the last fifty years.

1. Early Life, Education and Career

John was a thirteenth-generation American. His ancestor, Richard Mather came to New England on the ship *James* in 1635 and was the grandfather of the Puritan clergyman Cotton Mather.

John's father, Norman Mather, was a Princeton University professor of electrical engineering. John learned about mathematics from his father at an early age. While in high school, John attended junior-level mathematics courses at Princeton. He spent his undergraduate years at Harvard University, earning a bachelor's degree in 1964, and his graduate years at Princeton University, completing his Ph.D. in 1967. His advisor was John Milnor, who gave him the outstanding problem of trying to understand the structural stability of smooth mappings. Mather eventually succeeded in completely answering the question. His Ph.D. thesis,[1] the Mather-Malgrange preparation theorem, became the first installment of this monumental achievement.

In 1969, after two years at the Institut des Hautes Études Scientifiques, Mather was appointed associate professor at Harvard University. He was promoted to full professor in 1971. During this period Mather became a central figure in the development of the intertwined theories of stability of smooth mappings and of stratification theory.

In 1974 Mather accepted a visiting professorship at Princeton University, joining his father on the faculty. His position was converted to a full professorship the following year. In Princeton, he devoted decades to understanding the chaotic behavior of orbits of frictionless mechanical systems. Aside from sabbatical leaves, Mather remained at Princeton for the rest of his life.

Mather's mathematical writings reflected his meticulous attention to detail and exposition. He had a quiet and modest personality with a subtle sense of humor. He was supportive and generous to his students and freely contributed ideas to their work. As his colleagues noted[2] in their obituary for him, "John's behavior never betrayed the slightest hint that he was a distinguished man with many international honors. He was unfailingly self-effacing, scrupulously honest, and always willing to be of service to others."

Mather served for ten years on the editorial board of the *Annals of Mathematics* and for several decades on the editorial board of the series *Annals of Mathematics Studies* published by Princeton University Press. Mather received the John J. Carty Award for the Advancement of Science from the National Academy of Sciences in 1978 and became a member of the National Academy of Sciences in 1988. He received the Brazilian Order of Scientific Merit in 2000, the George David Birkhoff Prize in applied mathematics in 2003 and the Brouwer Medal from the Royal Dutch Mathematical Society in 2014. His name is attached to Aubry–Mather theory, Mather measures, Mather rings, the Thom–Mather isotopy lemma, the Mather–Thurston theorem on foliations and diffeomorphism groups, and the Mather connecting theorem.

2. Overview of Mather's work

John Mather was a world leader in two closely related subjects: (a) stability of smooth mappings and processes, and (b) singularities of spaces and mappings.

Subject (a) has its origin in engineering. Suppose we have a system that appears to be working correctly and predictably. Does a slight change in the initial conditions result in similar behavior (in which case we say the system is stable) or is it "chaotic" with wildly different behavior? How does an apparently stable system degenerate into a chaotic one? The same question may be asked of a smooth mapping. Do nearby mappings look the same, or are they wildly different? By "look the same" we could mean "up to smooth change of variables" or "up to continuous change of variables", resulting in the notions of *smooth* stability of mappings and *topological* stability of mappings respectively (see §3 and §4). Mather made fundamental contributions to all these questions. His first breakthrough, a monumental series of six remarkable papers, answered every conceivable question concerning smooth stability of mappings (cf. §3).

Subject (b) has its origin in differential and algebraic geometry: the set of points x where $f(x) = 0$ is an object which typically is smooth but in many interesting cases, may have crinkles, corners, cusps, and other singularities. Ever since Solomon Lefschetz[3] "planted the harpoon of topology into the whale of algebraic geometry," mathematicians have attempted to understand the nature and topology of the singularities that occur when the function f is algebraic or analytic. Is it possible that Cantor sets and fractals might arise as singularities in analytically defined subsets of Euclidean space?

Subjects (a) and (b) came together in the early 1960s when René Thom developed an audacious plan to understand the *topological* stability of smooth mappings (cf.§4). Thom's plan required a deeper understanding of the nature of the singularities of analytic sets than had been previously known (cf. §5). From 1967 to 1974, Mather orchestrated a remarkable exchange of techniques and results between these two areas. By 1975, he had turned Thom's vision into reality.

After arriving in Princeton in 1975, Mather's interests moved from stability to chaos (cf. §6). He spent decades discovering examples and developing techniques to understand the chaotic nature of orbits of frictionless mechanical systems, in which Cantor sets and fractal sets arise naturally. In the words[4] of his colleagues Charles Fefferman and David Gabai, "Mather's work completely bypassed earlier approaches to the problem, and exhibited an entirely unanticipated mechanism by which chaos arises. His theorems on this theme are among the deepest and most original mathematical results of the last fifty years."

3. Smooth Stability of Smooth Mappings

In 1955, Hassler Whitney[5] defined a smooth mapping f: $M \to P$ between smooth manifolds to be C^∞ *stable* if, for any sufficiently nearby mapping f': $M \to P$ there are diffeomorphism $\phi : M \to M$ and $\psi : P \to P$ that transform f' into f.

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "\phi"'] & P \arrow[d, "\psi"] \\ M \arrow[r, "f'"] & P \end{tikzcd}$$

Figure 1 C^∞ equivalence.

Whitney asked: Do stable mappings form an open and dense set in the space of all smooth proper mappings? For Morse functions and for mappings from the plane to the plane, Whitney showed the answer is "yes" and he was able to classify the smooth mappings. Whitney believed the answer would always be "yes."

In 1960 René Thom[6] dropped a bombshell: he found a smooth proper mapping $R^{16} \to R^{16}$ that cannot be smoothly approximated by stable maps. This opened a Pandora's box of complications.

Despite the efforts and partial results of many people, the answer was not known until 1968 when Mather[7–12] developed an enormous collection of techniques, described in six major papers that completely answered the question and in some sense killed the subject because there was nothing more to say.

Theorem 1. *Stable mappings $M \to P$ form a dense subset of the space $C^\infty(M, P)$ if and only if* $(\dim(M), \dim(P))$ *is represented by a red dot (as shown in Figure 2).*

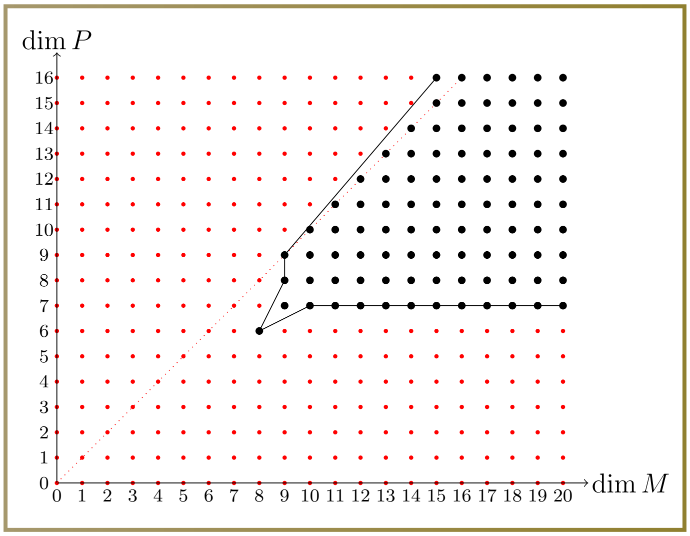


Figure 2 C^∞ stability range of dimensions.

4. Topological Stability of Smooth Mappings

During the 1960s, René Thom had been thinking that perhaps *topologically* stable maps might be dense, replacing the diffeomorphisms ϕ, ψ in Figure 1 with homeomorphisms. At first glance this might seem a hopeless task because a homeomorphism may be arbitrarily bad. But Thom developed a far-reaching vision as to how this might be achieved: (1) Develop a theory of stratifications that are *topologically* locally trivial along each stratum (see §5); (2) show that the jet space has a natural stratification in which the aforementioned families of orbits combine into finitely many strata; (3) and then show that transversality of the jet $J^k(f) : M \to J^k(M, P)$ to these strata, plus the "isotopy lemmas" imply that f is topologically stable. Thom's publication[13,14] of early versions of his proposal was followed by his monumental 1969 paper[15] that outlined the full ambitious program.

Mather explained[16] to the author that he had a great deal of difficulty in making sense of Thom's outline and he did not understand Thom's definition of the stratification of the jet space. In the end, using the notions of finite singularity type that he had developed in his papers on smooth stability, Mather was able to prove that topologically stable maps are dense. His proof differs from Thom's outline in that Mather does not use a stratification of the jet space, although he initially thought it would be necessary to do so. Mather published[17] an outline of his proof in 1973 but he was still hoping to write a book on the subject. Only the first chapter[18] was completed. This chapter, however, had an incredible influence on singularity theory and is still the best source on the foundations of stratification theory (cf. §5).

Mather later[19], citing his inspiration from Thom's outline, used his density theorem to construct a stratification of the jet space (different from Thom's). He showed that topological stability of $f : M \to P$ is implied by transversality of the jet mapping $M \to J^k(M, P)$ to this stratification, so it is a generic condition.

In 1976 Christopher Gibson[20] and colleages[21] published a second proof, that used many of Mather's techniques and results but followed Thom's outline more closely. They make use of a stratification of the jet space, possibly different from Mather's. They rely, in an essential way, on the foundational work on stratification theory in Mather's 1970 notes[18] which were available by then.

5. Stratification Theory

5.1. Stratifications. Can an analytic set exhibit fractal behavior? If $f : X \to Y$ is a proper analytic mapping between analytic sets, do the fibers fall into finitely many distinct homotopy types? These questions were partially put to rest when Stanislaw Lojasiewicz triangulated[22] semi-analytic sets. However, a triangulation of an analytic or algebraic set does not display the natural decomposition into more manageable pieces that these sets appear to have.

In 1947 Hassler Whitney[23] imagined dividing an algebraic set into a finite collection of smooth manifolds (or strata), that are glued together by some mysterious process. This would provide a notion of "equisingularity": two points in the same stratum have the same degree of singularity. Over the next twenty years, Whitney made various attempts[24–26] to refine these ideas. During the same period, notions of equisingularity were also pursued by Oscar Zariski.[27]

Eventually Whitney isolated two phenomena, which he illustrated with examples, that had been obstructing his approach to stratifications of algebraic sets. Whitney's first example is illustrated in Figure 3. It shows that the naive approach to stratify an algebraic set does not work. One might hope to start with the nonsingular part, then throw it away and continue by induction. In Whitney's example, the nonsingular part consists of the two-dimensional surfaces. When we remove this part, what remains is a smooth line.

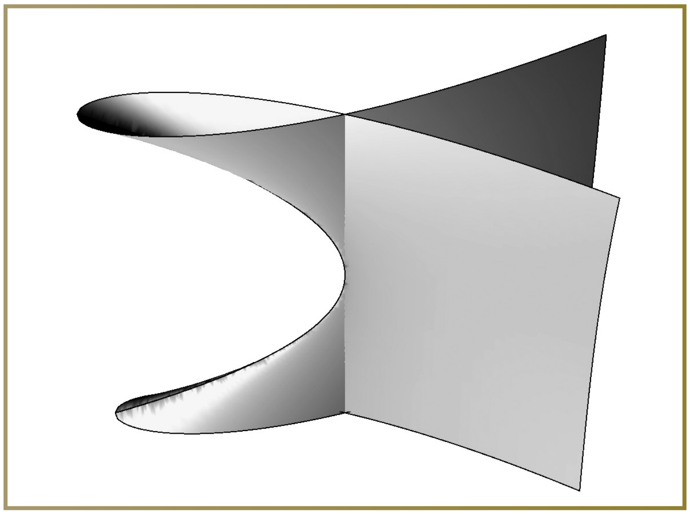


Figure 3 $y^2 = x^3 - x^2z^2$

But one point on this line is special, because of the way the two-dimensional part twists around the line. Whitney proposed conditions A and B as criteria that might (correctly) force the origin to be considered as a separate stratum. (It was later shown that algebraic, analytic, semi-algebraic, semi-analytic, subanalytic, and o-minimal sets admit stratifications satisfying Whitney's conditions A and B.)

Whitney's second example is more worrisome. It is an algebraic subset of Euclidean space such that no decomposition into smooth manifolds will be locally trivial in the C^1 sense. The variety shown in Figure 4 consists of four "sheets" meeting along the z axis. Three of the sheets are simply a product with the z axis, but the fourth sheet twists around the axis. Any differentiable flow in the ambient Euclidan space, parallel to the z axis, that preserves the first three sheets cannot preserve the fourth, because the derivative on the normal (two-dimensional) space, at any point on the z axis is determined up to a constant factor by the fact that it must preserve three lines. So, it cannot move the fourth line. Therefore, the homogeneity that is apparent in this example

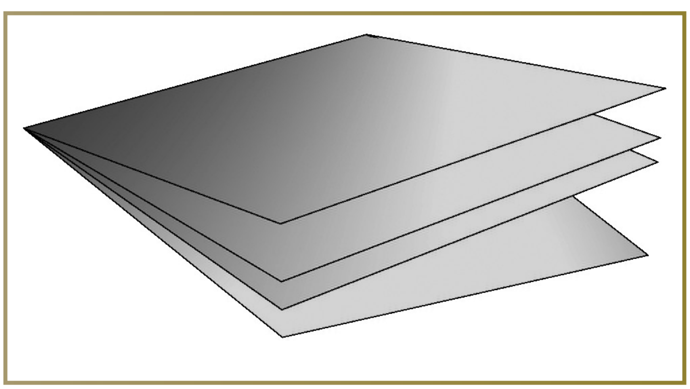


Figure 4 $xy(y - x)(y - zx) = 0$

can only be realized by a *continuous* but non-differentiable flow. It is this phenomenon (when it occurs in the space of jets) that is ultimately responsible for the fact that smoothly stable maps do not necessarily form a dense set in the space of all smooth mappings, but that topologically stable maps do form a dense set.

In his 1969 paper, as part of his plan to understand stability of mappings, Thom proposed a brilliant and ambitious program to solve both of Whitney's problems: a way to prove that a stratification satisfying Whitney's conditions would be *topologically* locally trivial. The plan is to construct extra data, tubular neighborhoods and distance functions for each stratum, and vector fields, smooth on each stratum, that respect this data. Although the vector fields are not continuous when strata come together, their flows join together to form a continuous flow.

5.2. Thom's outline[28] was difficult to follow and Mather believed[29] that the first step in proving his own results on topological stability required a rigorous proof that a Whitney stratified set is topologically locally trivial. Mather's 1970 notes,[30] the first chapter of his proposed book, accomplish this, following a modified version of Thom's outline, by providing precise definitions, careful estimates and double inductions.

5.3. One consequence of the theory is the following local structure theorem for stratified spaces. Let X be a stratum in a stratified space W. *Any point $x \in X$ has a basis of neighborhoods homeomorphic, by a stratum preserving homeomorphism, smooth on each stratum, to the product $B_x(\epsilon) \times c(L_x)$ where $B_x(\epsilon)$ is an open ball in X, and $c(L_x)$ is the cone over another stratified space, the link of the stratum X at $x \in X$.*

So, any stratification may be understood inductively in terms of the topology of the link. This key point is the basis for countless applications of the theory.

5.4. Mather's 1970 notes have had an enormous impact on the mathematical literature. Intersection homology theory and stratified Morse theory could not have been developed without Mather's foundational work. The Thom–Mather theory of stratifications has become a standard technique in algebraic geometry, symplectic topology, representation theory, and even in number theory. Although Mather's notes were not published until forty years after they were written and distributed, they have been cited hundreds of times and remain the single most complete and accessible approach to stratification theory. It is ironic that Mather's unpublished notes have had a greater influence on subsequent mathematical developments than his monumental work on the stability of smooth mappings.

6. Dynamical Systems

6.1. Mather's interest in the structural stability of smooth mappings is only one aspect of his more general interest in the stability of dynamical systems and diffeomorphism groups, in which even more complex behavior had been observed. After moving to Princeton in 1975 he began to concentrate on chaotic phenomena in dynamical systems.

6.2. Celestial Mechanics. Henri Poincaré had already understood that the Newtonian three body problem had chaotic behavior even if the solutions to Newton's equations remained smooth. In their chapter in Dynamical Systems, Theory and Application,[31] John Mather and Richard McGehee gave a surprising example of the Newtonian four body problem in which three of the bodies gang up on one, tossing it arbitrarily far, in finite time! Solutions to the four-body problem, with these particular initial conditions, simply do not exist beyond this critical time. (The initial conditions for which this disaster can occur form a Cantor set.) Such an alarming result leads us to question the stability of the Newtonian world and supports a famous 1897 conjecture of Painlevé[32] that the four-body problem exhibits singularities.

Their example is somewhat artificial in that the bodies lie on a single one-dimensional axis and collide with elastic collisions. Nevertheless, the example was very influential. Dmitry Anosov conjectured[33] that a non-collision example in two dimensions, sufficiently close to the Mather–McGeehee example might exist, but this has not yet been shown.

The unease generated by the Mather–McGeehee example was finally realized in 1992 when Zhihong Xia[34] found an example in $\mathbb{R}^3$ of five bodies under the action of Newtonian gravitation, that display unbounded solutions in finite time without collisions. (See also the excellent 1995 survey by Donald Saari and Zhihong Xia.[35])

6.3. Aubry Mather Theory of Twist maps. As Mather became more and more interested in chaotic systems, he turned his attention to the nature of the twist map. His work in this area has been highly influential. Twist maps of the cylinder arise from many different problems in dynamical systems: in solid state physics,[36] in the motion of a charged particle in a magnetic field (Fermi acceleration), in predator-prey dynamics, in the periodically forced pendulum, in the behavior of geodesics on a smooth manifold,[37] to name a few. Despite their apparent simplicity they are often the source of chaotic behavior in very complex systems. (See Theorem 8.3 in Golé,[38] Theorem 13.2.6 in Katok and Hasselblatt,[39] Moser's 1986 article,[40] or Sorrentino's book.[41])

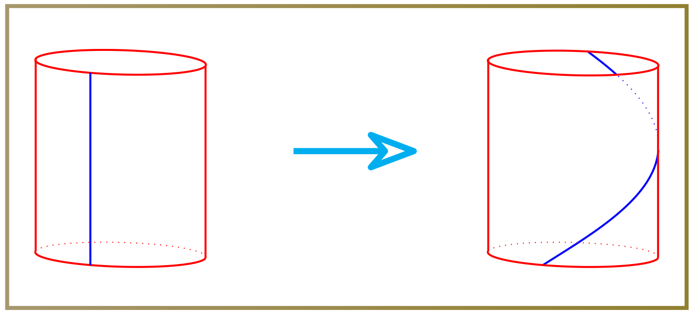


Figure 5 Twist map.

Let $I \times S^1$ denote a cylinder, where I is an interval $[a, b] \subset \mathbb{R}$ or possibly the real line itself, and where S^1 denotes the unit circle in the complex plane. A twist map $F(x, \theta) = (X(x, \theta), \Theta(x, \theta))$ is a diffeomorphism such that

(1) F is orientation preserving and area preserving
(2) the "ends" of the cylinder are preserved by F
(3) $\partial\Theta/\partial x > 0$.

Item (3) says that the image of each vertical segment "twists" around the cylinder in the counterclockwise direction, as shown in Figure 5.

Twist maps arise, for example, when considering the motion of a billiard ball on a billiard table with a smooth convex boundary, bouncing off the edge of the table with angle of incidence equal to angle of reflection, as described by Mather,[42] Golé,[43] Katok and Hasselblatt[44] or Moser.[45]

At such a bounce point p on the boundary, the incoming trajectory defines an angle, x with respect to the (positively oriented) tangent line at p, so $x \in [0, \pi] = I$. The position θ of the point p may be specified by the arclength from a basepoint to p, which is a number $\theta \in \mathbb{R}/L\mathbb{Z} = S^1$ where L denotes the total length of the boundary. The dynamics of the billiard ball then describe a twist map $F : I \times S^1 \to I \times S^1$.

If the billiard table is circular or elliptical then this motion is completely integrable, meaning that the trajectories of F lie on curves of constant energy in the phase space. Such a curve is invariant under the twist map F. The dynamics along such a curve may come from periodic orbits or they may be quite complicated when the angle θ is irrational.

The theorem of Andrey Kolmogoroff, Vladimir Arnold, and Jürgen Moser predicts that for small perturbations of an elliptical billiard table (and more generally, for small perturbations of any completely integrable Hamiltonian system) these invariant curves will survive. As the perturbation increases the invariant curves disintegrate. Aubry[36] and Mather[46] discovered, for arbitrary twist maps, that what remains are invariant Cantor sets on which a rotation number can still be defined. (The rotation number ρ of a homeomorphism f

John Mather and father. *With permission, Princeton University, Office of Communications.*

of the circle R/Z is the limit $\lim_{n\to\infty}(F^n(x) - x)/n$ where F : R → R is a lift of f. It is independent of x, is continuous as a function of f and is preserved under conjugacy of f.)

For the sake of simplicity, suppose the cylinder is finite with $I = [a, b]$ and suppose the twist map F preserves the ends of the cylinder. Let α, β denote the rotation numbers of the twist map F restricted to the two ends. The following statement summarizes some of the results of Aubry[47,48] and Mather.[49]

Theorem 2. *For every $\rho \in [a, b]$ there exists an invariant "Aubry–Mather" subset Γ_ρ of the phase space with rotation number ρ. It lies on the graph of a Liptschitz continuous function $g : S^1 \to I$. Every orbit in this subset has rotation number ρ. If ρ is rational then this subset is a circle and Γ_ρ contains periodic points. If ρ is irrational then every orbit is dense in Γ_ρ, and either Γ_ρ is the whole graph of g or it is a Cantor subset of the graph of g.*

Because of this last possibility, Aubry and Mather are said[50] to have found the "missing circles in KAM theory." In a two-dimensional system, the region between two invariant circles is necessarily preserved by the map F and such a region is "stable." But in the case of Aubry–Mather sets, orbits can escape through the gaps in the Cantor sets.

6.4. Arnold Diffusion. Suppose H is a completely integrable Hamiltonian function, defined on a four dimensional symplectic manifold M. The Hamiltonian flow preserves two independent action integrals $I_1, I_2 : M \to R$ whose simultaneous level set is typically a two dimensional torus. So the three-dimensional level surfaces of H are foliated by invariant tori.

Now consider a small perturbation of H. According to KAM theory many of these tori are preserved. The region between two of these tori is preserved by the flow (hence "stable") but is no longer foliated by invariant tori. If $\dim(M) > 4$ this argument fails, and Arnold conjectured that even for small perturbations, orbits exist on which one of the action integrals may change. He gave an example in his 1964 article.[51] This phenomenon came to be referred to as "Arnold diffusion" (even though it is not related to the usual notion of diffusion) because it represents a loss of stability. Kaloshin and Levi[52] give a long list of attempts to prove this conjecture.

In 2003, Mather announced[53] a proof of this conjecture. He worked very hard, over a period of years to complete this proof. In 2009 he gave a long series of lectures at the University of Maryland about his ideas to prove Arnold's Conjecture. For two months, he visited the University of Maryland each week, and gave an intense three-hour lecture. Although there were many essential missing ideas, Mather's notes and lectures became important ingredients in the final proof by Vadim Kaloshin and Ke Zhang.[54] Mather passed away before the book appeared, but he took great satisfaction in following its development.

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