BIOGRAPHICAL MEMOIRS

Hassler Whitney

March 23, 1907–May 10, 1989
Elected to the NAS, 1945

A Biographical Memoir by John W. Morgan

Hassler Whitney was a topologist extraordinaire. He began his mathematical work in the 1930s during an extremely active period in both algebraic and geometric topology. He made many important contributions that remain fundamental and foundational today. Also, as he pointed out in his summary of an influential conference on topology in Moscow in 1935, he was part of the wave of American mathematicians who were responsible for a shift in the center of topology from Europe to the United States. Most of these mathematicians were trained in Europe and fleeing the horror of the coming war. Whitney was one of the few fathers of this American flowering who was born in this country. Later there were many more.

Let me mention, by way of introduction, several of Whitney's most enduring contributions. Today, it is well-known that cohomology with its ring structure is a much more powerful invariant than the homology groups. Whitney was the first to push the importance of cohomology as opposed to homology, and the multiplication of this ring structure is induced by the Alexander-Whitney diagonal formula. Obstruction theory as introduced by Whitney has extended far beyond the simple examples he gave to become a fundamental tool throughout algebraic topology. The paradigm that Whitney established for the Stiefel-Whitney classes consisting of naturality under pullback of bundles and the Whitney sum formula for direct sums of bundles is the model for the Chern classes and the Pontrjagin classes. And of course there is the "Whitney trick." It is the central reason that all of surgery theory and the h-cobordism theorem are valid for manifolds of dimensions ≥ 5, but don't hold for dimensions less than 5.

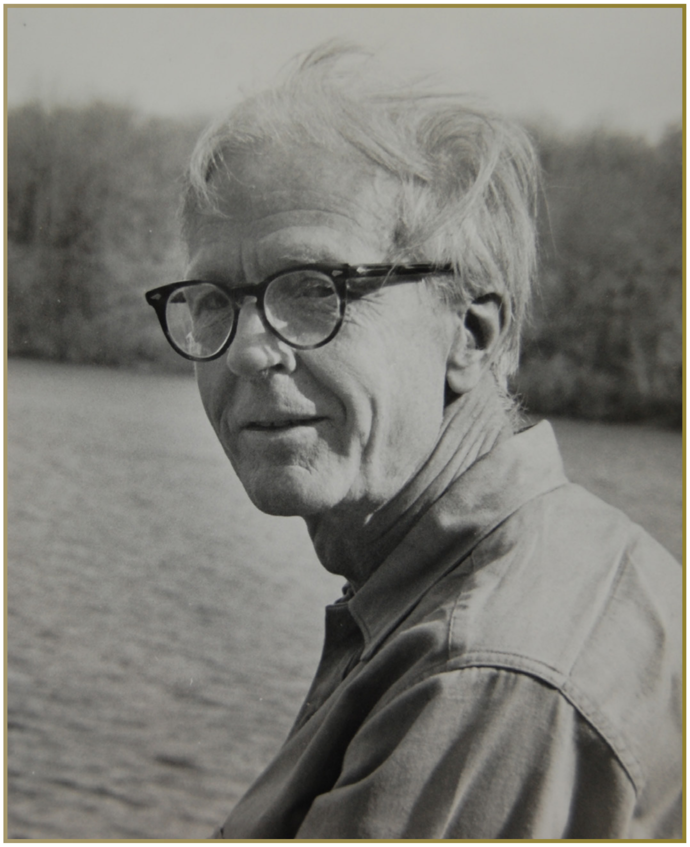

Figure 1 Hassler Whitney in 1973. *Photo by Sally W. Thurston.*

Whitney introduced stratified spaces into geometric topology and in particular formulated essential conditions, Whitney's Conditions A and B, which are natural and ubiquitous and are central in the program importing the techniques from differential topology of smooth manifolds to study stratified spaces. Whitney's extension theorems give necessary and sufficient conditions for functions defined on subsets of $\mathbb{R}^n$ to extend to C^m-functions on all of $\mathbb{R}^n$.

This is by no means a complete list, but it gives some of the high points of Whitney's contributions.

In addition to mathematics, which we will examine in more detail below, Whitney's other passions were mountaineering and music. Late in life, after observing a granddaughter's grade school education in math, Whitney took a keen interest in math education, writing and lecturing on the subject.

Early Life and Education

Hassler Whitney was born on March 23, 1907, in New York City into an illustrious family that included governors and state supreme court justices as well as artists and academics. His father, Edward Baldwin Whitney, was a New York Supreme Court judge, and his mother, A. Josepha Newcomb Whitney, was an artist and political activist. Hassler lived in Switzerland for two years, from fourteen to sixteen, and his love of the Alps and of climbing became lifelong. The famous photo of Whitney at age fourteen atop a spire in the Alps shows his fearlessness. He climbed with noted topologists John Alexander and Georges de Rham. In the United States, he is known for the first ascent of the Whitney-Gilman Ridge on Cannon Cliff, New Hampshire, in 1929, a climb done without protection (pitons) but with ropes.

Whitney attended Yale University, graduating in 1928 with a bachelor's degree in physics, and stayed for another year to complete a degree in music. He was an accomplished performer on the violin and viola. He attended graduate school at Harvard University and earned a Ph.D. in mathematics in 1932 with a thesis entitled "The Coloring of Graphs," written under the direction of George David Birkhoff. During his time at Harvard, he married Margaret Howell on May 30, 1930, and the couple would have three children: James Newcomb, Carol, and Marian. From 1931-33, he was a National Research Council fellow at Harvard and then Princeton University. In 1935, he was hired as an assistant professor at Harvard, rising to full professor by 1952. That year, he joined the Institute for Advanced Study at Princeton as a professor of mathematics and remained there for the rest of his life, becoming an emeritus professor in 1972. He married twice more: Mary Garfield on January 16, 1955, with whom he had two children (Sarah Newcomb and Emily Baldwin), and Barbara Floyd Osterman on February 8, 1986. Whitney died on May 10, 1989, in Princeton. His ashes rest atop the Dents Blanches in the Alps.

Whitney was a member of the National Academy of Sciences, the London Math Society (honorary), the Swiss Math Society (honorary), the Académie des Sciences (foreign associate) and the American Philosophical Society. Among his many honors, Whitney received the U.S. National Medal of Science in 1976, the Wolf Prize in Mathematics in 1982, and the Steele Prize from the American Mathematical Society in 1986.

Whitney and Algebraic Topology

Algebraic topology, and in particular homology theory, was one of the hottest topics in mathematics in the 1930s. To summarize Whitney's work on homology, the first thing to say is that he pushed the idea that cohomology was an important object of study and in many contexts was more appropriate than homology. He is responsible for defining the ring structure in cohomology. Whitney introduced obstruction theory as a tool for studying maps up to homotopy. Using obstruction theory he defined what are now called the Stiefel-Whitney characteristic classes of a sphere bundle.

During the 1930s, there was a concerted effort by mathematicians to understand the nature of homology, with many different versions of homology being developed and many geometric and topological applications being found. Whitney brought his own perspective to this effort. He firmly believed that cohomology gave a different and, at times, more powerful point of view than homology. In fact, in the debate over its appropriate name, he pushed for use of the term *cohomology*, as opposed to the more functorially correct *contra-homology*. His suggestion prevailed. In addition to giving cohomology its name, he made several seminal advances in cohomology theory that are central to all of algebraic topology today.

One of his first uses of cohomology[1] was to give a different and more direct proof of a then newly established result by Paul Alexandroff and Heinz Hopf in classifying maps up to homotopy from an n-dimension complex K to S^n.[2] As Whitney realized, the result is simpler to state using cohomology, because the description in terms of homology involves both the n^{th} homology of K and the torsion of its $(n-1)^{st}$ homology.

Whitney showed that associating to a map $f: K \to S^n$ the pullback under the map induced by f on cohomology of the fundamental class of S^n produced a bijection between the homotopy classes of such maps $K \to S^n$ and $H^n(K;\mathbb{Z})$. His proof of this bijection was the first and simplest example of a very general technique, *obstruction theory*, for studying maps between spaces up to homotopy. Obstruction theory doesn't usually lead to a complete answer to classification up to homotopy as it did in this example, but it always gives a first approximation to the answer. It remains one of the basic tools of algebraic topology.

James Alexander and Andrey Kolmogorov made identical proposals for a product on cochains, during the 1935 Moscow International Mathematics Conference. As Whitney pointed out, these couldn't be correct because using them gave a product whose dimension was one less than the sum of the dimensions of the factors. Motivated by this, Whitney modified the proposal to the one we use today.[3] Simultaneously, Eduard Čech realized the analogous necessary modification in his cohomology theory.[4] Alexander immediately saw the value of their formula over his and incorporated it into his theory.[5] What Whitney and Čech produced is known today as the *Alexander-Whitney* (or sometimes simply the *Whitney*) diagonal map. At the most basic level it sends a simplex $\langle v_0, \ldots, v_n \rangle$ to

$$\sum_{i=0}^{n} \langle v_0, \ldots, v_i \rangle \times \langle v_i, \ldots, v_n \rangle$$

(The difference from the Alexander-Kolomogorov proposal is in the repetition of the vertex v_i in this formula.)

Replacing the cross product by a tensor product and extending linearly produces a chain map $C_*(X) \to C_*(X) \otimes C_*(X)$, which produces an associative co-multiplication. (Depending on the context, $C_*(X)$ is either the singular chain complex of a topological space X or the simplicial chains on a simplicial complex X.) The algebraic dual to this map is an associative multiplication on the algebraically dual complex of cochains, inducing an associate multiplication on cohomology. Miraculously, the induced multiplication on cohomology is commutative with the usual sign encountered throughout algebraic topology, though this is obviously not true for the cochain product. (A complete understanding of this point had to await Norman Steenrod's development of his squares, which measured in a precise way the non-commutativity on the cochain level of this multiplication.[6] Later developments also clarified that, although one can make other choices for the diagonal formula, any choice that satisfies a simple, geometrically natural condition gives the same multiplication on cohomology.)

This fundamental structure for cohomology was more support for Whitney's belief that cohomology was a powerful alternative to homology, because associating to a space its cohomology ring gives more information than just associating its homology or cohomology groups. Today this structure is omnipresent and fundamental in algebraic topology.

Whitney's Characteristic Classes

Whitney defined characteristic cohomology classes for the tangent bundle of a manifold, for normal bundles of manifolds in a larger manifold, and more generally, for sphere bundles over arbitrary complexes with structure group the orthogonal group.[7] Simultaneously, Eduard Stiefel had defined homology classes for tangent bundles of manifolds.[8] Whitney's theory was more general in that it covered a much wider class of examples. It also was formulated in terms of cohomology rather than homology, which is necessary when considering bundles over bases that are complexes rather than manifolds. Today, just as Whitney presented things, the characteristic classes, called the *Stiefel-Whitney classes*, of a vector bundle E over a complex K are cohomology classes, denoted $w_i(E) \in H^i(K;\mathbb{Z}/2\mathbb{Z})$, with the total Stiefel-Whitney class

$$W(E) = 1 + w_1(\mathrm{E}) + w_2(\mathrm{E}) + \cdots \in H^*(K;\mathbb{Z}/2\mathbb{Z}).$$

The definition Whitney gave was again as an obstruction theory class. Let K be a simplicial complex and E an n-dimensional vector bundle. Using general position, choose $n - r + 1$ linearly independent sections $\{s_1, \ldots, s_{n-r+1}\}$ over the $(r - 1)$-skeleton of K. For each r-simplex σ, the obstruction modulo 2 to extending these to $n - r + 1$ to linearly independent sections over σ gives the value of $\mathbb{Z}/2\mathbb{Z}$-cochain $w_r(E, s_1, \ldots, s_{n-r+1})$ on σ. This cochain is a cocycle, and varying the sections $\{s_1, \ldots, s_{n-r+1}\}$ over $(r - 1)$-skeleton varies the cocycle by a coboundary. Its cohomology class it the r^{th} Stiefel-Whitney class. (Whitney also understood that the odd classes, $w_3, w_5, \ldots$ are naturally integral classes when the bundle is oriented.)

From a modern point of view, we say that given an n-dimensional vector bundle E over a complex K, it always has $n - r + 1$ linearly independent sections (linearly independent at each point) over its $(r - 1)$-skeleton. There is an obstruction theory for extending these to linearly independent sections over the entire complex, and the first obstruction is

$$w_r(E) \in H^r(K; \pi_{r-1}(V(n - r + 1, \mathbb{R}^n))),$$

where $\pi_{r-1}(V(n - r + 1, \mathbb{R}^n))$ is the r^{th} homotopy group of $V(n - r + 1, \mathbb{R}^n)$, the Stiefel manifold of $n - r + 1$ linearly independent vectors in $\mathbb{R}^n$. (If the bundle is non-orientable, then the coefficients are twisted by tensoring with the orientation of the bundle.) Easy computations show that the first non-zero homotopy groups of the Stiefel manifolds are

$$\pi_{r-1}(V(n - r + 1, \mathbb{R}^n)) = \begin{cases} \mathbb{Z}/2\mathbb{Z} & \text{for r} \equiv 0 \pmod 2 \\ \mathbb{Z} & \text{for r} \equiv 1 \pmod 2 \end{cases}.$$

Thus, Whitney's classes $w_r(E)$ are the first non-trivial obstruction to extending the $(n - r + 1)$ linearly independent sections over the $(r - 1)$ skeleton to all of K, and, as always with the first non-trivial obstructions, they are well-defined independent of the choices over the lower skeleta.

Whitney showed that these classes are natural under pullback of bundles. In addition, he showed that they satisfy the basic result he called a duality theorem for his characteristic classes, a result we now call the *Whitney sum formula*. Namely, if E_1 and E_2 are vector bundles over K then

$$W(E_1 \oplus E_2) = W(E_1) \cup W(E_2).$$

In addition to establishing this result, Whitney went on to sketch a proof (which he never published in detail) of a conjecture of Stiefel's that, for a compact triangulated manifold M^n, the sum of the i-simplicies of the first barycentric subdivision is a cycle whose homology class is Poincaré dual to the $w_{n-i}(TM)$, where TM is the tangent bundle of M.

Today there are three sets of characteristic classes that satisfy the two conditions that Whitney laid out for his classes, namely naturality under bundle maps and a cup product formula for the class of a direct sum of vector bundles. In addition to the Steifel-Whitney classes, they are the Pontrjagin classes for orientable vector bundles (where the Whitney sum formula is valid only modulo 2-torsion) and Chern classes for complex vector bundles, both of the latter lying in integral cohomology. (The Chern classes can be viewed as obstructions to extending complex linearly independent sections over skeleta; the Pontrjagin classes of a real bundle are defined in terms of the Chern classes of its complexification.)

Whitney Embedding Theorem and its Consequences

Whitney also made fundamental contributions to the topology of smooth manifolds. His original insight with the largest effect on the topology of manifolds goes under the name of the "Whitney trick." This result is at the heart of the fact that the study of manifolds divides into dimensions ≥ 5, dimension ≤ 4. The higher dimensions can be handled by exclusively topological techniques and more or less all dimensions at once, 4-periodically by dimension. The lower dimensions, 3 and 4, are much more difficult and need extra techniques from geometry and partial differential equations. In fact, we do not have nearly the understanding of 4-manifolds that we do of manifolds in all other dimensions.

The Whitney trick arose from his reaction to the then new "abstract" definition of a manifold in terms of an atlas of coordinate charts and overlap functions. Whitney asked himself if this definition produced any new manifolds that we didn't already know about; that is to say, manifolds that are not smooth submanifolds of Euclidean space. He quickly answered that question in the negative by using a countable atlas of coordinate charts and a partition of unity to embed the manifold in $\mathbb{R}^\infty$ (or $\mathbb{R}^N$ for some large N if M is compact).

He then asked himself the follow-up question: Given n, what is the smallest k so that, up to diffeomorphism, every n manifold is a submanifold of $\mathbb{R}^k$? He showed that given an n-dimensional manifold M^n in $\mathbb{R}^{k+1}$, for the general linear projection $\mathbb{R}^{k+1} \to \mathbb{R}^k$, the restriction of the projection to M^n gives an embedding as long as $k \geq 2n + 1$. This leads to Whitney's first (or weak) embedding theorem.[9]

Theorem 1. *Any* n*-dimensional manifold embeds in* $\mathbb{R}^{2n+1}$.

He went on to ask, Is this the best we can do? His response, and in particular the method of proof, is what set the stage for the amazing developments in the classification of manifolds during its glory days, from roughly 1960 through 1975.

Theorem 2. (*Whitney's Strong Embedding Theorem*) *For every* $n > 2$, *any* n*-dimensional manifold* M^n *embeds in* $\mathbb{R}^{2n}$.

First, we need only consider connected manifolds. In brief, one uses the first Whitney embedding theorem to embed M^n in $\mathbb{R}^{2n+1}$ and takes a generic linear projection to $\mathbb{R}^{2n}$. The result is an immersion of M^n in $\mathbb{R}^{2n}$ with a discrete set of double points where two sheets cross like two generic n-dimensional affine linear subspaces in $\mathbb{R}^{2n}$. If the manifold is closed and orientable, homological intersection theory shows that there must be an even number of intersection points, each intersection has a sign, and the algebraic sum of these signs is zero, so that algebraically at least the intersection points cancel. In general, near any point of self-intersection we can introduce a simple local change, an n-dimensional "loop," that creates a new point of self-intersection algebraically cancelling the given intersection point. The problem is then to remove a pair of algebraically cancelling self-intersections.

Now comes the *Whitney trick*.[10] Let $\{x, y\}$ be a pair of algebraically cancelling self-intersections for M^n immersed in $\mathbb{R}^k$ where $k = 2n$, one plus and one minus sign of intersection. Then on each of the two sheets draw arcs between these points to form a loop in $\mathbb{R}^{2n}$ meeting each sheet in these arcs. This loop bounds a disk, a *Whitney disk*, which since $n > 2$, we can assume is embedded and disjointed (except along its boundary) from the image of M^n. We then push a neighborhood of one of the arcs in M across the Whitney disk from its position in the original sheet to the other sheet and then continue the deformation slightly to the other side of that sheet. This removes the two points x, y of intersection without creating any new points of intersection. Continuing in this way deforms the immersion of M^n to an embedding in $\mathbb{R}^{2n}$.

(It turns out that by direct construction, not using the Whitney trick which does not hold in general for surfaces in 4-manifolds, one can embed every surface in $\mathbb{R}^4$. In fact the strong embedding theorem holds for all n.)

Whitney was working with smooth manifolds, explicitly using techniques from differential topology. But when the dust settled, it turns out that his trick also works in the piecewise linear category and, after much work, in the topological category as well.

The Whitney trick allows one in great generality to cancel, that is, deform so as to remove, points of intersection between sub-manifolds of an ambient manifold that cancel algebraically (have the opposite sign) as long as the ambient dimension is at least 5 and the complement of the union of the two submanifolds is simply connected. The simply connected hypothesis and the dimension constraints are the ones needed to ensure the existence of Whitney disks, which are used to cancel pairs of intersection points that have an opposite sign.

The use of Whitney's idea of using 2-disks (now called "Whitney disks" to cancel excess points of intersection led to a revolution in the study of manifolds. The first revolutionary use of this technique was by Stephen Smale in his proof of the *h*-cobordism theorem for simply connected manifolds of dimension at least 5.[11] (The statement is that two compact manifolds are diffeomorphic if they are connected by a compact manifold that is homotopy equivalent to a product.) He easily reduced the problem to the case in which the connecting manifold has handles only of two adjacent indices, say r and $r + 1$, near the middle dimension. The remaining step was to cancel in pairs the remaining handles. It is an elementary algebraic argument to arrange that the boundaries of the $r +$ 1-dimensional descending handles algebraically cancel the ascending manifolds of the r-dimensional handles. Smale invoked the Whitney trick to realize the algebraic intersection without excess "cancelling" pairs of intersection points between the ascending manifolds from the critical points of index r and the descending manifolds of the critical points of index $r + 1$ and completed the proof of the *h*-cobordism theorem.

This was the beginning of surgery theory. John Milnor and Michel Kervaire, William Browder, Sergei Novikov, Charles T. C. Wall, and many others developed high-dimensional (i.e., dimensions ≥ 5) surgery theory, which led to classification of such "high-dimensional" manifolds in terms of homotopy theory.[12–15] The reason that one needs the lower bound on the dimension both in the *h*-cobordism theorem and in surgery theory is that one repeatedly uses the Whitney trick to eliminate excess geometric intersections between submanifolds. This is the only step that truly requires the dimension assumption.

The lower dimensions, 3 and 4, require completely different techniques. For example, Michael Freedman used incredibly delicate, purely topological arguments to establish the Whitney trick for topological 4-manifolds and thereby established that Smale's argument for *h*-cobordism theory extends to cover simply connected topological (as opposed to piecewise linear or smooth) 4-manifolds.[16] In contrast to Freedman's result, Donaldson almost simultaneously used new invariants inspired by gauge theory in physics to show that the *h*-cobordism theory does not hold for smooth and piecewise linear manifolds. One consequence of this marked dichotomy between smooth and topological manifolds is that $\mathbb{R}^4$ has more than one, in fact uncountably many, non-diffeomorphic smooth structures. For all other n, $\mathbb{R}^n$ has a unique smooth structure up to diffeomorphism.

The situation in dimension 3 is another story because things are completely controlled by the fundamental group, as Grigori Perelman showed[17,18,19] early in the twenty-first century. He used Ricci Flow theory developed by Richard Hamilton[20] in the space of Riemannian metrics on the 3-manifold. In particular, just as Henri Poincaré had conjectured in 1905, up to diffeomorphism, there is only one compact simply connected smooth 3-manifold: the 3-sphere. Because every topological or piecewise linear (pl) 3-manifold has a smooth structure unique up to diffeomorphism, the original 3-dimensional Poincaré Conjecture holds in all three categories: smooth, pl, and topological.

Whitney Stratified Spaces

Whitney's work on stratified spaces remains the foundation of the study of these spaces. We begin with Whitney's general definition. A *stratification* of a space X is a partition of X into locally closed subsets $\{X_i\}_{0\leq i\leq N}$ where for each i the subspace X_i is an i-dimensional manifold. The connected components of the X_i are called the *strata* of the stratification. One requires that the frontier of any stratum is a union of strata of lower dimension. A *stratified space* is a topological space with a stratification. In studying stratified subspaces of $\mathbb{R}^n$, Whitney introduced conditions that he called Conditions A and B.[21,22] Suppose that P^k and Q^ℓ are smooth submanifolds of $\mathbb{R}^n$. The pair is said to *satisfy Condition A* if for every sequence of points p_i in P converging to a point q_i in Q any limit in the Grassmannian of k-planes in $\mathbb{R}^n$ of the sequence of tangent planes $\{T_{p_i}P\}_i$ is a k-plane containing the tangent plane T_qQ. The pair is said to *satisfy Condition B* if for every $\{p_i\}$ in P and $\{q_i\}$ in Q converging to the same point $q \in Q$ with the straight lines ℓ_i through p_i and q_i converging to a line ℓ (automatically through q), the line ℓ translated to the origin is contained in the tangent plane T_qQ. (In fact, John Mather proved that Condition B implies Condition A.[23]) Although Whitney's original definition was for P, Q smooth submanifolds in Euclidean space, it is immediate that the notion generalizes directly to pairs of smooth submanifolds in any ambient smooth manifold. A stratification of a subset of a smooth manifold M *satisfies Whitney's Conditions A and B* or is a *Whitney stratification* if every pair of strata satisfy these conditions.

Whitney stratifications are quite useful because, first, many naturally occurring geometric objects support such a stratification and, second, they were extremely powerful conditions allowing one to import the usual techniques from differential topology of smooth manifolds to the category of these stratified spaces by working inductively over the strata.

As to the first point, Whitney proved any complex algebraic variety in $\mathbb{C}^n$ or any complex analytic variety in $\mathbb{C}^n$ satisfies Conditions A and B. The result is somewhat subtle, as was demonstrated by the example called *Whitney's Umbrella*, which is the complex algebraic subvariety X given given by $\{u^2 - v^2z = 0\}$ in $\mathbb{C}^3$. Whitney showed that the obvious stratification, given by X_2 equal to the z-axis and X_4 is the complement of the z-axis in X, is a stratification with smooth strata, but it does not satisfy Conditions A and B (at the origin). To achieve a stratification that satisfies Conditions A and B, one needs to take X_0 to be the origin, X_2 the complement of the origin in the z-axis, and X_4 as before.

As to the second point, the Whitney conditions allow one to do transversality in a manner similar to the case of smooth manifolds. If $X \subset M$ is Whitney stratified and $\varphi : N \to M$ is a smooth map of smooth manifolds, then there is an arbitrary small perturbation of φ in the C^∞-topology that is transverse to each stratum of X, and consequently $\varphi^{-1}(X) \subset N$ has an induced stratification satisfying Conditions A and B.

Whitney's result was later extended to real semi-algebraic sets and sub-analytic sets by René Thom and Heisuke Hironaka, respectively.[24,25]

Abstract Stratified Spaces

Motivated by Whitney's Conditions A and B for stratified subsets, Thom[26] and Mather[27] introduced an abstract notion of a stratified space (not necessarily a subspace of a smooth manifold) now called a *Thom-Mather stratification*. This is a stratification with *control data*, namely a stratified space in which strata have smooth structures and, additionally, each stratum F has an open neighborhood $T(F) \subset X$ with a retraction $\pi_i : T(F) \to F$ and a function $\rho_F : T(F) \to [0, \infty)$ such that $\rho_F^{-1}(0) = F$. Furthermore, for any pair of strata $F \neq G$ with $F \subset \overline{G}$, we have

1. $(\pi_F, \rho_F) : T(F) \cap G \to F \times (0, \infty)$ is a smooth submersion,
2. for any $x \in T(F) \cap T(G)$, we have $\pi_G(x) \in T(F) \cap G$ and $\pi_F \circ \pi_G(x) = \pi_F(x)$ and $\rho_F(\pi_G(x)) = \rho_F(x)$ for all $x \in T(F) \cap T(G)$

Two sets of smooth structures for the strata and control data for a stratified space are *equivalent* if the smooth structures on each stratum agree and if the two sets of control data have a common restriction to smaller control data.

Any stratified subspace of a smooth manifold satisfying Conditions A and B has compatible control data and hence becomes a Whitney stratification. Furthermore, just as for Whitney's Conditions A and B, the Thom-Mather control data allows one to import the tools of differential topology, for example, a version of Sard's theorem for Whitney stratifications.

Today, Whitney's Conditions A and B and the Thom-Mather stratification are fundamental tools in the study of singular spaces, including algebraic, semi-algebraic, and sub-analytic varieties. One application of these ideas is the proof of the theorem, conjectured (with an outline of a proof) by Thom and completed by Mather,[28] that for smooth manifolds M and N, the topologically stable maps in $C^\infty(M, N)$ are dense in the Whitney C^∞-topology.

Integration

Whitney studied manifolds from both a combinatorial point of view (such as triangulations and dual cell structures) and a smooth point of view of differential topology, and he produced deep and lasting results from each approach. His book *Geometric Integration Theory* is a bridge between these two points of view.[29] Whitney starts by considering the space of polyhedral chains in $\mathbb{R}^n$ and defines the *flat* norm of a k-chain A by

$$|A^\flat| = \inf_D |A - \partial D| + |D|,$$

where D ranges over all polyhedral chains of degree $k + 1$. A *flat* k-cochain is a linear function X on polyhedral k-chains for which there is $N < \infty$ with

$$|X(A)| \leq N\,|A^\flat|$$

for every polyhedral k-chain A. The infimum of such N is the *norm* of X. Whitney showed that the completions of these normed spaces are dual Banach spaces, respectively closed under the boundary operator ∂ on polyhedral chains and adjoint on flat cochains. Furthermore, he showed that there is a symmetric multiplication (up to the usual signs) of flat cochains. Through a standard limiting process, Whitney shows that a flat cochain has a well-defined value on smooth k-chains, and this value is invariant under smooth changes of coordinates. This leads to a theory of flat cochains on smooth manifolds.

In the space of measurable differential forms, there is an analogous notion of flat. Whitney invokes a result that he attributes to John H. Wolfe that flat cochains are defined by integrating measurable flat k-forms and that integration determines an isomorphism of the Banach space of flat forms and flat cochains.[30] This allows a proof of Georges de Rham's theorem through this intermediary: flat cochains evaluate to give real-valued cochains on a smooth triangulation and measurable forms include ordinary forms, with both inducing isomorphisms on cohomology. The theory of flat cochains and flat forms informed Dennis Sullivan's approach to rational homotopy theory via piecewise polynomial forms on simplicial complexes.[31]

Whitney Extension Theorems

We now describe Whitney's early work on extension of functions. Given a set $E \subset \mathbb{R}^n$, a continuous function $f : E \to \mathbb{R}$, and a positive integer m, when is f the restriction of a C^m function $F : \mathbb{R}^n \to \mathbb{R}$ to E? There are two flavors of questions:

1. E is an open set, and
2. E is arbitrary.

As an example of the difficulties that can arise in questions of flavor 1, consider a branch of the analytic function $f(z) = z^{21/2}$ on the domain

$$E = \{z \in \mathbb{C} \mid |z| < 1, -\pi + |z|^{100} < \arg(z) < \pi - |z|^{100}\}.$$

The function f and its derivatives up to order 10 are uniformly continuous on E, yet f does not extend to a C^1-function on the complex plane. In response, Whitney introduced the chord-arc criterion, which states that there exists a constant C such that any two points $x, y, \in E$ can be joined by a curve in E of length at most $C|x - y|$. Suppose E is an open set satisfying the chord-arc condition, that f is a C^m-function on E, and that f and its derivatives up to order m are uniformly continuous on E. Whitney proved that under these assumptions f extends to a C^m-function on $\mathbb{R}^n$. The study of smooth extensions of functions from open sets to all of $\mathbb{R}^n$ continues to generate deep mathematics.[32]

Now let us consider questions of flavor 2, the extension of functions from arbitrary sets $E \subset \mathbb{R}^n$ to all of $\mathbb{R}^n$. Without loss of generality, we can take E compact. Consider $f : E \to \mathbb{R}$ with $E \subset \mathbb{R}^n$ compact. We ask when for fixed m there exists a function $F \in C^m(\mathbb{R}^n)$ with $F = f$ on E. Whitney gave a complete answer in the one-dimensional case. His solution involved "divided differences," that is, natural generalizations of difference quotients. The case $n > 1$ is significantly harder. Its solution came only in the twenty-first century, as noted in Charles Fefferman's 2006 discussion of this subject.[33] But as early as 1934, Whitney solved a variant of flavor 2 by proving the *Whitney extension theorem*.[34] That result answers the following question: Fix $m, n \geq 1$ and let $E \subset \mathbb{R}^n$ be compact. Suppose that to each point $x \in E$ we assign an m^{th} degree polynomial P^x on $\mathbb{R}^n$? How can we tell whether there exists a function $F \in C^m(\mathbb{R}^n)$ whose m^{th} order Taylor polynomial at each $e \in E$ agrees with P^x?

Whitney's extension theorem asserts that the obvious necessary conditions on the P^x arising from Taylor's theorem are sufficient for the existence of the desired F, which can even be taken to be a real analytic away from E. As an application, Whitney produced a remarkable example of a C^1 function on $\mathbb{R}^2$ whose set of critical points contains an arc on which the function is non-constant.[35] Whitney's proof of the Whitney extension theorem is based on a simple, fundamental idea, namely a recursive procedure that partitions the complement of E into "Whitney cubes" with favorable geometric properties. In particular, the diameter of each Whitney cube is comparable to its distance from E.

The idea of producing a family of pairwise disjoint cubes by means of a recursive procedure reappeared in the 1950s as the Calderon-Zygmund decomposition.[36] Calderon-Zygmund and Whitney families of cubes continue to play a significant role in analysis.

Matroids

Whitney's early work, including his thesis, was on graph theory and the theory of colorings of graphs. This led him to the definition of a *matroid* as a finite set together with a collection of subsets, called *distinguished subsets*, of the set satisfying certain axioms.[37] One example is when the set is the set of edges of a graph and the distinguished subsets are the edges forming a forest (that is, containing no loops). Another example is a finite subset of a vector space with the distinguished sets being subsets of the given set that are linearly independent. Although these objects do not play the same central role in topology as the Whitney cup product, or his obstruction theory, or his characteristic classes, they have turned up recently in several disparate contexts in combinatorics and topology.

Matroids have color polynomials generalizing the color polynomial of a graph. The color polynomial of a matroid is related to the colored Jones polynomial of knots.[38] There is a recent deep result that shows that the absolute values of the coefficients of the color polynomial of a matroid are log convex. This might be related to an old conjecture due to Fox that the absolute values of the Alexander polynomial of a knot are log convex.

In another direction, matroids also play a prominent role in the Gelfand- MacPherson construction of a combinatorial formula for the Pontrjagen classes of a smooth manifold in terms of a smooth triangulation.[39]

Summary

This survey shows the broad sweep of Whitney's work in topology and geometry and highlights the continuing centrality of his work in these fields. In reading Whitney's papers, one senses an almost playful approach to mathematics. Whitney worked by asking natural questions, then mulling over simple examples in an effort to clarify what is at issue, and then generalizing from the understanding of these examples to discover general important principles of wide significance. This approach gives his work a refreshing naturalness that makes it a pleasure to read his papers.

Note

All of the early biographical data on Whitney is taken from Keith Kendig's excellent biography *Never a Dull Moment:*

Hassler Whitney, Mathematical Pioneer (Providence, R.I.: MAA Press, 2018). Also, the section in this memoir on Whitney Extension Theorems was written by Charles Fefferman.

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